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Multi-unit auctions with uncertain supply and single-unit demand*

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May 8, 2023

Abstract

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1 Introduction

Commodities and financial instruments are often traded in multi-unit auctions. For example, treasury auctions in the U.S. and wholesale electricity markets around the world are cleared by a uniform-price auction. Most treasury auctions in Europe, on the other hand, use pay-as-bid auctions. Pay-as-bid auctions are often used when procuring ancillary services for electric-power systems.[1] The traded volume is uncertain in wholesale electricity markets and in some treasury auctions.[2] In this paper, we compare the two auction formats, taking into account that information can be asymmetric and that the traded volume can be uncertain. We find that uniform-pricing is better for the auctioneer, as long as equilibria are well-behaved. We also identify circumstances where the auctioneer benefits from disclosing its information.[3]

A concern with uniform-price auctions is that they can result in outcomes with prices at the collusive level. This is known from several theoretical papers, such as Wilson (1979), Klemperer & Meyer (1989), von der Fehr & Harbord (1993) and Back & Zender (1993). In practice, seemingly collusive equilibria in uniform-price auctions have for example been observed in the procurement of electric-generation capacity (Schwenen, 2015) and in fishing-quota auctions (Marszalec et al., 2020). Another of our contributions is that we identify circumstances for which well-behaved equilibria can be ensured in the uniform-price auction.

We prove our results by extending the single-object settings studied by Milgrom & Weber (1982) and Blume & Heidhues (2004) to a multi-unit setting, where the traded volume of the auctioneer can be uncertain. Similar to them, we consider a sales auction, but results are analogous for procurement auctions. To make progress, we make the simplification that bidders have single-unit demand, i.e. each bidder buys at most one good.[4] Hence, the market is competitive in the sense that strategic demand reduction is not an issue. Similar simplifications of multi-unit auctions have been made by Vickrey (1961), Milgrom (1981) and Weber (1983).

[1] Ancillary services are for example used to control the frequency and voltage in the power system.

[2] In electricity markets, producers submit offers before the level of demand and amount of available production capacity are fully known. In this case, the traded volume of strategic producers is uncertain due to demand shocks and intermittent output from non-strategic, renewable energy sources. In Mexico, Finland and Italy, the treasury sometimes reduces the quantity of issued bonds after the bids have been received (McAdams, 2007). In treasury auctions in U.S. there is often an uncertain amount of non-competitive bids from many small non-strategic investors (Wang and Zender, 2002; Rostek et al., 2010). IPOs sometimes incorporate the so-called "Greenshoe Option", which allow issuing firms to increase the amount of shares being offered by up to 15% after the bids have been submitted (McAdams, 2007).

[3] This is for example relevant for European electricity markets. EU regulations have improved transparency of these markets during the last years (Lazarczyk & Le Coq, 2019; von der Fehr, 2013). In March 2023, EU proposed that transparency of electricity markets should increase even further.

[4] Each bidder could have a limited storage capacity, be liquidity constrained or have preferences, such that at most one unit would be bought. In an analogous procurement auction each bidder has capacity to produce at most one unit of the good.

Each bidder receives private information in the form of a signal and it is used to estimate the value of the traded good. In our setting this value could be correlated with the auctioneer's supply. We allow the auctioneer to also receive a private signal, which it can choose to disclose to all bidders. In practice, this could for example be aggregated bid data from previous auctions.

We start by studying symmetric equilibria for bidders that are symmetric ex ante, before private information has been received. Values of bidders are interdependent. This means that a bidder has an estimate of the good's value, the bidder's signal, and that it would get a (weakly) better estimate if it could also observe the signals of the competitors and the auctioneer. As an example, if a bidder wants to buy treasury bonds, then the bidder would get a better estimate of the value of the bond if it also had access to the estimates made by other market participants. The signals of bidders and the auctioneer are assumed to be affiliated[5]. The information structure is similar to Milgrom & Weber (1982), but we allow the number of sold units, Z, to be larger than one and uncertain. We allow signals and Z to be correlated if signals and Z jointly satisfy a certain affiliation property. Many of the results in Milgrom & Weber (1982) can be generalized as long as this property is satisfied.

We show that there is a well-behaved, monotone, symmetric equilibrium in both the uniform-price and pay-as-bid auction, which is efficient.[6] For the special case where signals are independent of Z, we are able to solve for all symmetric equilibria. We show that non-monotonic bid functions can be ruled out in the uniform-price auction and that there is exactly one symmetric equilibrium; a well-behaved equilibrium. For well-behaved symmetric equilibria, we show that an auctioneer will find it beneficial (its expected revenue weakly increases) to ex ante disclose its own signal. This is sometimes referred to as a publicity effect. Moreover, we find that uniform pricing gives a (weakly) higher revenue for the auctioneer in comparison to pay-as-bid pricing. The ranking result and the publicity effect for the pay-as-bid auction are proven by means of a linkage-principle argument that we have generalized to a multi-unit auction with single-unit demand and uncertain supply.

Related results have been found by Weber (1983) and Holmberg & Wolak (2018). Our contribution to the literature on symmetric equilibria is that we allow for uncertain supply of the auctioneer, and multiple bidders with signals that could be correlated with the supply. In addition, we allow for (and rule out) non-monotonic bid functions.

A problem with uniform-price auctions is that ill-behaved equilibria can exist. We study this issue in detail, and how such equilibria can be prevented. To make progress, we simplify the information structure by assuming that each bidder has full information of its valuation of the good, but is not informed of competitors' valuations. This corresponds to bidders having private values. Values could be

[5] Affiliation is a strong version of positive correlation. One implication of the affiliation property is that if the observed signal of a bidder increases, then, conditional on this increase, the expected value of the other signals will also increase.

[6] McAdams (2003) establishes existence of a monotonic pure-strategy equilibrium in the uniform-price auction when bidders have multi-unit demand.

correlated, but we assume that the support of the probability density of a bidder's value is independent of Z and values of competitors. These are restrictive assumptions, but the model is less restrictive in other aspects. We no longer require that signals/values are affiliated. We do not make any assumptions on the joint probability distribution of the values of the bidders and the auctioneer, except for some regularity conditions. The joint probability distribution isn't necessarily common knowledge for the bidders. It is sufficient that the support of the signals/values is common knowledge. Bidders can be asymmetric ex ante. We solve for all pure-strategy Bayesian Nash equilibria, including asymmetric equilibria.[7]

We show that equilibria with prices at the collusive level can exist in uniform-price auctions, even if we assume single-unit demand so that bidders have limited market power. We refer to them as high-low equilibria, because some bidders always bid high and others always bid low. The high bids are always accepted and the low bids are always rejected. The highest bid among the low bids sets the price. For the case with at least two objects and at least two more bidders than objects, we show that any uncertainty in the auctioneer's supply will knock out high-low equilibria in uniform-price auctions. If all bidders have the same value range, then there is a unique equilibrium, which is well-behaved, if there is any doubt what the auctioneer's supply will be.

If supply is certain, then an effective price floor or price cap,which is binding with a positive probability, will give a unique equilibrium. Hence, there are circumstances where introducing a maximum price can actually increase the revenue of an auctioneer that is selling goods, by eliminating the high-low equilibrium. Single-object second-price auctions are special.[8] In this case, an effective price floor gives uniqueness (Blume & Heidhues, 2004), but we find that a price cap does not ensure uniqueness. Another special case is when there is exactly one more bidder than the number of objects. In this case it is the other way around. An effective price cap gives a unique equilibrium, but not a price floor.

For the case with at least three bidders, at least two objects and at least two more bidders than objects, we also study a setting where bidders have different value ranges. If bidders have different upper bounds on values, then bidders can play a high-low equilibrium for the highest values (a, b), at the top of the value range and above the upper bound of some bidders. A bidder observing a value in that range always bid high (at b) or low (at a). Uncertainty in the auctioneer's supply, or a price floor, is not enough to get rid of this partial high-low equilibrium. An effective price cap (or elastic supply at high prices) is necessary to get a unique equilibrium. If bidders have different lower bounds on values, then bidders can play a high-low equilibrium at the bottom of the value range, below the lower bound of some bidders. The high-low equilibrium at the bottom is removed by an effective price floor (or elastic supply at low prices). If bidders have different

[7] Our methodology would allow us to also characterize all mixed-strategy equilibria. But this would make the presentation more complicated and technical. Also, allowing for such equilibria would not change our conclusions for the private-value model in any substantial way. Hence, we decided to exclude such equilibria from the analysis.

[8] We consider a uniform-price auction with single-unit demand, where the price is set by the highest rejected bid. In the single-object case this corresponds to a second-price auction.

bounds both at the top and bottom, then both an effective price floor and an effective price cap are needed to get a unique equilibrium. When unique equilibria occur they are well-behaved and efficient.

Our analysis of uniform-price auctions shows that this auction format has an invariance property. If there is an equilibrium in an auction with n bidders and Z goods, then there is a corresponding equilibrium in a transformed auction with n bidders and $n - Z$ goods, if the signs of all values and bids are reversed. We refer to the transformed auction as the reflected auction. As an example, the reflected version of a single-object, sales auction with n bidders is a sales auction with n bidders and $n - 1$ goods. This explains why a price floor is needed to get uniqueness in the single-object auction and a price cap in the reflected auction.

It is well-known that positive affine transformations do not alter the set of equilibria. Together with the property of the reflected auction this significantly simplified some of our proofs. Moreover, the property makes our results easier to understand intuitively. We have not seen similar results in the previous auction literature, but in spirit this result is somewhat related to game transformations, game isomorphisms, best-response equivalence and strategic equivalence in games studied by Thompson (1952), Kohlberg & Mertens (1986), Harsanyi & Selten (1988), Elmes & Reny (1994) and Morris & Ui (2004).

The remainder of the article is organized as follows. Section 2 explains our contribution to the auction literature in greater detail. Our symmetric, uniform-price model with affiliated values is introduced in Section 3. In Section 4, we show that this model has exactly one symmetric equilibrium, and then we characterize this equilibrium. In Section 5, we extend the linkage principle to a multi-unit auction with single-unit demand and uncertain supply. In Section 6 we study an auction with pay-as-bid pricing, and how it compares with the uniform-price auction. In Section 7, we analyse the private-value model, which allows for asymmetric bidders. Section 8 summarizes our main results. The proofs and some technical lemmas are in the Appendix.

2 Contribution to the auction literature

In this section we discuss our contribution relative to the previous literature in greater detail. In many ways our work is closest to the single-object auction models by Milgrom & Weber (1982) and Blume & Heidhues (2004). Our main contribution relative to them is that we allow the auctioneer's supply to be larger than one and uncertain. Moreover, Milgrom & Weber (1982) do not consider non-monotonic bid functions and Blume & Heidhues (2004) do not consider interdependent values. In addition, in the private value case, we allow bidders to have different value ranges. As will be shown, these differences have a major impact on the set of equilibria.

2.1 Multi-unit auctions

The focus of our study is uniform-price auctions (with some comparisons to pay-as-bid auctions) where bidders have either private values or affiliated signals and interdependent values. Each buyer is only interested in buying one (indivisible) unit of the good, i.e. single-unit demand. We consider a uniform-price auction, where the highest rejected bid (the first rejected bid) sets the clearing price. This has similarities to single-unit demand settings studied by Vickrey (1961), Milgrom (1981) and Weber (1983). Demand reduction and its associated inefficiencies are not issues for our model.

Uniform-price auctions with private values and multi-unit demand have for example been studied by Anderson & Holmberg (2018) and Burkett & Woodward (2020A, 2020B). The Vickrey-Clarke-Groves (VCG) auction is a generalisation of the Vickrey auction to multi-unit demand, such that truthful bidding is preserved as a weakly dominant strategy for private values. VCG auctions with private values have for example been studied by Blume et al. (2009) and Reny & Perry (1999).

We study multi-unit auctions with a single bidding round. Dynamic multi-unit auctions with sequential bidding are for example studied by Ausubel (2004) and Donald et al. (2006).

2.2 Divisible-good auctions

Von der Fehr & Harbord (1993) consider divisible-good, uniform-price auctions with two bidders that have symmetric information. Bids are restricted to be flat, so each bidder can only choose one bid price. Holmberg & Wolak (2018) introduce asymmetric information into such a model.[9] Divisible-good auctions with flat bids are somewhat similar to a multi-unit auction model with single-unit demand. Still there is a difference to our model, as a bid can be partially accepted for divisible goods. A partially accepted bid sets the clearing price in a uniform-price auction, both for the accepted quantity of the partially accepted bid and the accepted volume of competitors. Hence, the divisible-good, uniform-price auction with flat bids is actually a combination of uniform pricing and pay as bid. This makes the analysis of divisible-good auctions more complicated, which explains why results in Holmberg & Wolak (2018) are less clean compared to Milgrom & Weber (1982). Moreover, even if bids are flat, the set of equilibria are often quite different in a setting with divisible goods and single-unit demand.

Wilson (1979), Ausubel et al. (2014) and Vives (2010;2011) consider divisible-good auctions with asymmetric information.[10] Unlike us, they allow each bidder to use a demand function to condition its trade on the price, which corresponds to multi-unit demand. Wilson (1979) and Back & Zender (1993) argue that low-revenue equilibria is a bigger problem in divisible-good auctions with non-flat bids than in corresponding multi-unit auctions with single-unit demand, but this

[9] Fabra et al. (2006) extend the model by von der Fehr & Harbord (1993) to pay-as-bid auctions. Fabra and Llobet (2023) analyse uniform-price auctions where producers are privately informed of production capacities. Wolak (2007), Kastl (2012) and Holmberg et al. (2013) analyse divisible-good auctions where each producer can submit several flat bids.

[10] Rostek & Weretka (2012) generalize the model of Vives (2011) to double auctions where both buyers and sellers are strategic.

argument is based on an analysis of symmetric equilibria. As explained in the next subsection, asymmetric low-revenue equilibria can occur in uniform-price auctions with single-unit demand.

2.3 Low-revenue equilibria

Blume & Heidhues (2004) solve for all equilibria in a second-price auction with at least three bidders that have private independent values in a common range $\left[0, v^h\right]$. Each equilibrium in the second-price auction can be characterised by a threshold $\widehat{b}$. An equilibrium exists for every $\widehat{b}$ in the value range $\left[0, v^h\right]$. For values below $\widehat{b} \in \left(0, v^h\right)$, a single bidder bids high (at $\widehat{b}$) and others bid low (at 0). All bidders bid at value above the threshold. Hence, the single bidder wins the object and pays nothing if all bidders observe values below the threshold. We refer to this as a high-low equilibrium at the bottom of the value range.[11] In the degenerate case where $\widehat{b} = 0$, one gets a well-behaved equilibrium, where all bids are at value and the allocation of the object is efficient. In the other degenerate case where $\widehat{b} = v^h$, one gets a high-low equilibrium, where a single bidder always bids high (at v^h) and others always bid low (at 0), irrespective of values. Hence, the single bidder always win the auction and the revenue of the auctioneer is always zero.

Blume & Heidhues (2004) consider a one-shot game without any tacit collusion. Still, the high-low equilibrium has similarities to a collusive bidding ring with bid rotation (Hendricks & Porter, 1989), where the date rather than signals would decide whose turn it is to bid high. The existence of this type of non-cooperative equilibrium implies that a collusive agreement would be self enforcing once it has been established, so that the ring would be stable.

The reflected version of a single-unit auction with n bidders has n bidders and $n-1$ goods. We prove that also the equilibria are reflected. Hence, except for the degenerate cases where $\widehat{b} = 0$ or $\widehat{b} = v^h$, every equilibrium in the auction with n bidders and $n-1$ goods is a high-low equilibrium at the top of the value range. For values above $\widehat{b}$, a single bidder bids low (at $\widehat{b}$) and others bid high (at v^h). If all have values above $\widehat{b}$, then the bid at $\widehat{b}$ is rejected. All the other bids are accepted and the market price is $\widehat{b}$. All bid at value for values below the threshold. We refer to high-low equilibria at the top or bottom as partial high-low equilibria.

In both the single-object auction and the reflected auction, there is a single bidder that bids at $\widehat{b}$ with a positive probability. It turns out that these are special cases. Partial high-low equilibria do not exist in uniform-price auctions with single-unit demand if at least two objects are traded, at least two bids are rejected (there are at least two more bidders than objects) and bidders have values in a common range $\left[0, v^h\right]$.[12] The reason is that if at least two bidders bid at $\widehat{b} \in \left(0, v^h\right)$ with a positive probability, then one of them would find it profitable to deviate and bid

[11] Blume et al. (2009) refer to this as the first-class of equilibria. They solve for similar equilibria in the VCG auction.

[12] Partial high-low equilibria can exist in multi-unit auctions, if bidders have multi-unit demand. In this case, a single bidder can bid at the threshold $\widehat{b}$ with a positive probability for several of its units. Blume et al. (2009) prove this for a VCG auction.

slightly higher than $\widehat{b}$. Equilibria with several bidders bidding at $\widehat{b}$ with a positive probability only exist for the degenerate case, where bids at $\widehat{b}$ are either rejected with certainty or accepted with certainty as in the high-low equilibrium.

Another finding in this paper is that, even if at least two objects are traded and at least two bids are rejected, partial high-low equilibria can occur if bidders have heterogenous value ranges. For sufficiently high prices, above some bidders' upper bound of values, one will reach a range where at most one of the bids in that range will be rejected. High-low equilibria at the top will exist in that range. For sufficiently low prices, below the lower bound of some bidders, one will reach a range where at most one of the bids in that range will be accepted. High-low equilibria at the bottom will occur in that range.

von der Fehr & Harbord (1993) find equilibria that are related to the high-low equilibrium in a divisible-good, uniform-price auction with flat bids. But as a partially accepted bid is price setting, such equilibria can only exist if one bid is partially accepted and all other bids are fully accepted. Hence, a bidder with the partially accepted bid is pivotal. If this bidder would leave the auction, then the auction would be undersubscribed. This is related to Burkett & Woodward (2020A) who consider non-pivotal bidders and find that letting the last accepted bid set the uniform price in a multi-unit auction will remove outcomes with prices at the collusive level.

An example where uniform-price auctions have failed miserably in practice is the procurement of capacity in New York's electricity market (Schwenen, 2015),[13] which was often cleared at the collusive level.[14] Several electricity markets in the U.S. and South America have had similar problems when using uniform-price and VCG designs to procure capacity.

2.4 Unique equilibria

Several papers find that there are unique equilibria if bids are restricted to be non-decreasing with respect to signals.[15] Blume & Heidhues (2004) go one step further. They allow for non-monotonic bid functions. For three (or more) bidders with private independent values, they show that there is a unique equilibrium, if

[13] The purpose of the capacity payment is to give electricity production a subsidy. This increases the production capacity in the market, which lowers the risk of having blackouts.

[14] Marszalec et al. (2020) find related problems in the fishing-quota auctions in the Faroe islands.

[15] Bikhchandani & Riley (1991;1993) find unique equilibria in the second-price auction for common and interdependent values, respectively, when bid strategies are restricted to be increasing and continuous with respect to signals. Lizzeri & Persico (2000) prove uniqueness for a general class of two-bidder, single-object auctions when strategies are non-decreasing with respect to signals, or if signals are independent. Their result does not apply to second-price auctions, which they view as a knife-edge case in the large class of mechanisms. Uniqueness in first-price auctions has been studied in detail by Maskin & Riley (2003) and McAdams (2007).

Holmberg & Wolak (2018) restrict bids to be non-decreasing with respect to signals and prove uniqueness in uniform-price and pay-as-bid auctions with flat bids for two bidders that have an uncertain pivotality status.

the reservation price is effective.[16] This means that the reservation price is binding with a positive probability. Bidders observing sufficiently low signals will not take part in the auction.

Similar to Blume & Heidhues (2004) we consider multiple bidders. We allow bid functions to be non-monotonic and discontinuous, but we require bids to be regular. By this we mean that bid functions are piecewise continuous, where each piece is constant, monotonically increasing or monotonically decreasing. For the private-value model of a uniform-price auction, we show that a tiny uncertainty in the auctioneer's supply is sufficient to get a unique equilibrium if bidders have identical value ranges. Related uniqueness results have been proven for divisible-good auctions with symmetric information (Klemperer & Meyer, 1989; McAdams, 2007; Anderson & Hu (2008); Anderson, 2013; Holmberg, 2008) and multi-unit auctions with multi-unit demand and private values (Anderson & Holmberg, 2018). However, in those studies where bidders have significant market power, large uncertainties in the traded volume are needed to get a unique equilibrium.

If supply is certain, then an effective price floor or price cap can give a unique equilibrium in our model, but there are exceptions. In a single-object auction, an effective price floor, but not a price cap, gives uniqueness. It is the other way around in the reflected version of the single-object auction, which has exactly one more bidder than the number of objects. For the case with at least three bidders, at least two objects and at least two more bidders than objects, we also study the case where bidders have different value ranges. If bidders have different upper bounds on values, then an effective price cap is necessary to get a unique equilibrium. If bidders have different lower bounds on values, then an effective price floor is necessary to get a unique equilibrium. If bidders have both different lower and upper bounds, then both an effective price floor and an effective price cap are needed to get a unique equilibrium.

To reduce the risk of getting prices at the collusive level in the procurement of electric-generation capacity, Harbord & Pagnozzi (2014) have suggested that the volume traded by the auctioneer should be partly random, which is in line with our findings. One issue with capacity auctions is that prices have been very volatile. Some auctions, such as the procurement of capacity by the system operator in New York (NYISO), have introduced price floors to stabilize prices, which reduces the risk to investors in production capacity.

Schwenen (2015) studies the effect that such price floors have on outcomes with prices at the collusive level in New York. During 2006/2007, the price cap was at 150% of the net cost of new entry (net CONE) for a gas-fired plant. During the period that Schwenen observes he finds 55 outcomes with prices at the collusive level. In counterfactual simulations, he shows that a bid floor at 75%, 50% and 25% of net CONE would reduce the number of occurrences with prices at the collusive level by 37, 23 and 10, respectively. In the capacity market of NYISO, seemingly collusive outcomes are to great extent driven by dominant suppliers of capacity. They increase the price by strategically withholding capacity and

[16] Blume et al. (2009) find similar results for VCG auctions. Burkett and Woodward (2020B) find related results for a uniform-price auction with multi-unit demand.

offering it at the cap. Such suppliers sometimes find it profitable to withhold capacity even if the price floor is above the marginal cost for all of the offered capacity. This is not the case in our model of a sales auction, where bidders have single-unit demand. Hence, bid floors and price caps need to be more restrictive in a model where bidders have market power, if seemingly collusive outcomes are to be avoided.

Many capacity markets in the U.S. have tried to improve the procurement of capacity by making the system operator's demand elastic with respect to the price (Aagard & Kleit, 2022). In our model, with single-unit demand, it is as effective for the auctioneer to buy back one unit of the good at low prices as to have a price floor at the same price.[17] Similarly, the auctioneer could have price-sensitive supply at high prices instead of a price cap. This might be attractive for an auctioneer if it could find an extra unit to sell when the price is high. Otherwise, it would be costly for an auctioneer to have elastic supply at high prices.

We have not been able to prove uniqueness of equilibria for the affiliated-signals model. But if we consider ex-ante symmetric bidders with bidding strategies that are regular and assume that the auctioneer's supply is independent of signals of the bidders, then we can show that there is a single symmetric equilibrium, which is well behaved. As a special case, this result also applies to second-price auctions.

2.5 Efficiency and ex-post optimality

Similar to Weber (1983) and Milgrom & Weber (1982), we find that symmetric equilibria of uniform-price and pay-as-bid auctions with ex-ante symmetric bidders are efficient, at least with respect to aggregated information among market participants.[18] In general, these equilibria are not ex-post optimal. [19] In the special case with private values and single-unit demand in a uniform-price auction, we show that all equilibria are almost surely ex-post optimal. These equilibria are robust, and do not depend on the probability distribution of signals nor on risk aversion of the bidders. For private values, the high-low equilibrium and partial high-low equilibria are also ex-post optimal, but not efficient. This result has some parallels in Birulin (2003), who finds inefficient ex-post optimal equilibria in English auctions.

[17] More elasticity is beneficial for the auctioneer if bidders have significant market power (LiCalzi and Pavan, 2005).

[18] Anderson and Holmberg (2018) establish efficient symmetric equilibria for a multi-unit, uniform-price auction with private values and multi-unit demand. Holmberg & Wolak (2018) show that efficient equilibria can be found in a uniform-price, divisible-good auction with asymmetric information if bids are required to be flat.

Ausubel et al. (2014) identify a special case where efficiency does occur in a divisible-good, uniform-price auction with asymmetric information, but in general such an auction would be inefficient.

[19] To get fully ex-post optimal outcomes for interdependent values, one would normally need a dynamic auction format, such as an English auction in the case of a single-object auction and a dynamic VCG mechanism (Ausubel, 2004; Donald et al., 2006) for multi-unit auctions.

2.6 Comparisons of uniform-price and discriminatory auctions

The linkage principle essentially says the following: The more closely the winning bidder's payment is linked to its signal, the greater the expected revenue will be for the auctioneer. The linkage principle was first introduced by Milgrom & Weber (1982) and the intuition behind this concept was further developed by Krishna (2010) and Quint (2014). In this paper, we extend the linkage-principle argument further, so that it can be used to rank revenues in multi-unit auctions with single-unit demand when the number of traded units is correlated with the signals of bidders. We use the linkage principle to show that an auctioneer would (weakly) prefer uniform to discriminatory pricing, if the bidders' signals and the auctioneer's supply satisfy the necessary affiliation conditions and bidders play the symmetric equilibrium. Uniform pricing and pay-as-bid pricing are revenue equivalent when signals are independent and the auctioneer's supply and signals are independent.[20] Weber (1983) find the same ranking results when the auctioneer's supply is certain. Holmberg & Wolak (2018) prove related results for a divisible-good auction with two bidders and a restrictive information structure. Ranking of auction formats becomes more complicated for multi-unit demand when strategic demand reduction comes into play (Ausubel et al., 2014; Baisa & Burkett, 2018).

Most previous comparisons of uniform-price and discriminatory auctions have been made for models where information is assumed to be symmetric among bidders. These studies tend to conclude that pay-as-bid pricing is better for the auctioneer (Holmberg, 2009; Pycia & Woodward, 2021; Fabra et al., 2006).[21] There are several possible explanations for why these studies rank the auction designs differently to us. One is that we consider asymmetric information. Moreover, we assume single-unit demand, while Holmberg (2009) and Pycia & Woodward (2021) consider strategic demand reduction. Another explanation for differences in the ranking results is that Pycia & Woodward (2021) and Fabra et al. (2006) consider less well-behaved equilibria among the set of equilibria in the uniform-price auction, while, when making comparative statics, we assume that bidders play the well-behaved, symmetric equilibrium. The latter could be the case due to uncertainty in the auctioneer's traded volume, an effective price floor/cap, or because bidders have coordinated to this equilibrium.

Empirical studies by Armantier & Sbaï (2006;2009) and Hortaçsu & McAdams (2010) find that the treasury would prefer uniform pricing in France and Turkey, respectively, whereas Kang & Puller (2008) find that discriminatory pricing would be best for the treasury in South Korea. Results for the U.S. treasury are incon-

[20] If the auctioneer's supply is correlated with the signals and the affiliation conditions are satisfied, then the auctioneer will weakly prefer a uniform-price auction.

Weber (1983) finds that pay-as-bid pricing is better than uniform-pricing for the auctioneer, if symmetric bidders are risk averse and have values that are private and independent.

The revenue equivalence result corresponds to the results that Myerson (1981) and Riley & Samuelson (1981) proved for first- and second-price auctions.

[21] Pycia and Woodward (2019) consider small asymmetries in bidder's information to show that their results are robust to small changes in their information structure.

clusive (Archibald & Malvey, 1998).

2.7 Publicity effect

For circumstances where bidders play a well-behaved symmetric equilibrium and the bidders' signals and the auctioneer's supply satisfy the necessary affiliation conditions, we show that it is beneficial for the auctioneer to disclose its signal. This is true for both the uniform-price and pay-as-bid auction. Moreover, we find that it is better for the auctioneer to always fully disclose its signal, than to disclose it partially or to disclose it sometimes, for example depending on the value of its signal. This is in line with the results on the publicity effect that were proven by Milgrom & Weber (1982) for single-object auctions. Holmberg & Wolak (2018) prove similar results for a duopoly market with a restrictive information structure.[22] Another related paper is Vives (2011), who finds that mark-ups decrease when non-pivotal bidders receive less noisy cost information before competing in a uniform-price, procurement auction.

Our model is different to previous work in that a bidder's signal could be informative of the auctioneer's supply. We find that in expectation, an auctioneer does not gain anything from disclosing information that helps bidders to predict the value of the good in a pay-as-bid and uniform-price auction, unless the information also helps bidders to predict competitors' signals or sales of the auctioneer. An example of the latter is Pycia & Woodward (2021) who study a divisible-good, uniform-price auction where bidders have symmetric information and where disclosing the auctioneer's supply increases the revenue of the auctioneer.

The publicity effect does not always hold in multi-unit auctions where each bidder can buy more than one unit. Perry & Reny (1999) present such a counter example for a VCG auction. Moreover, increased transparency could increase the risk of collusion in a repeated multi-unit auction (von der Fehr, 2013).

3 Affiliated-signals model

The model with affiliated signals is general in how valuations depend on the signals. In particular, the valuation of a bidder is allowed to depend on the signals of other bidders. In order to make progress with this model, we make, similar to Milgrom & Weber (1982) and Weber (1983), several restrictive assumptions. We assume that signals are affiliated, that bidders are symmetric ex ante (before signals are received) and that they have single-unit demand and are risk-neutral.[23] Another similarity with Milgrom & Weber (1982) is that we only study pure-strategy equilibria that are symmetric ex ante. Asymmetric equilibria will be considered in Section 7.

[22] The information structure was particularly restrictive for the uniform-price auction, where Holmberg and Wolak (2018) established the publicity effect when bidders had independent signals. The publicity effect was driven by the fact that the signal of the auctioneer was correlated with bidders' signals.

[23] Milgrom and Weber (1982) and Weber (1983) consider risk-neutral bidders in their main analysis, but they have extensions where bidders are allowed to be risk averse.

Our analysis extends Milgrom & Weber (1982) in that we allow a bidder's bid function to be non-monotonic with respect to signals. Also, we allow the auctioneer to sell more than one unit and we allow the auctioneer's supply to be uncertain. We let Z be the number of items that are auctioned, with $\underline{Z} \leq Z \leq \overline{Z}$. In the case that $\underline{Z} < \overline{Z}$, Z may take more than one value, and Z is unknown to the bidders. Moreover we assume that each value Z with $\underline{Z} \leq Z \leq \overline{Z}$ has a positive probability, irrespective of signals observed by bidders. Our assumptions on $\underline{Z}$ and $\overline{Z}$ are assumed to be common knowledge among bidders.

Each bidder $i \in \{1, \ldots, n\}$ receives a private signal X_i which has information about the value of an object. Similar to Milgrom & Weber (1982) one could think of the signal as a value estimate.[24] We allow the signal to be correlated with Z. Hence, it could potentially be used to predict Z. Let $\boldsymbol{X}$ be a vector with all private signals and let $\boldsymbol{X}_{-i}$ be a vector with all private signals except for X_i. In addition, we let $\boldsymbol{S} = [S_1, \ldots, S_m]$ be a vector with m signals, which are informative of the value of the good.[25] Sometimes we find it convenient to write $\widetilde{\boldsymbol{X}}$ for the set of affiliated signals, (excluding X_1), $\widetilde{\boldsymbol{X}} = \{S_1, ..., S_m, X_2, ...X_n\}$. The value $V_i = u_i(\boldsymbol{S}, \boldsymbol{X})$ of the object to bidder i will depend on all signals, including signals that are not observed by the bidder. Note that the value does not directly depend on Z. We make the following standing assumptions for the affiliated-values model:

Assumption 1: There is a function u on $\mathbb{R}^{m+n}$ such that for all i, $u_i(\boldsymbol{S}, \boldsymbol{X}) = u(\boldsymbol{S}, X_i, \boldsymbol{X}_{-i})$, where u is symmetric in its last $n-1$ arguments. Hence, all of the bidders' valuations depend on $\boldsymbol{S}$ in the same manner, and bidders valuations are symmetric with respect to the private signals.

Assumption 2: The function u is non-negative, is continuous and non-decreasing in its variables, and is strictly increasing in X_i.[26]

Assumption 3: For each bidder i, the expected value $\mathbb{E}[V_i]$ conditioned on a subset of signals S and X is defined (i.e. bounded) no matter what subset is chosen.[27]

In our model $f(\boldsymbol{S}, Z, \boldsymbol{X})$ denotes the joint probability density of the signals and Z. We can condition on the possible values of $Z \in \{1, 2, ..., n-1\}$ and we write $f_k(\boldsymbol{S}, \boldsymbol{X})$, $k = 1, 2, ..., n-1$, for the joint probability density of the signals conditional on $Z = k$. We assume that $f_k(\boldsymbol{S}, \boldsymbol{X})$ is continuous with respect to $\boldsymbol{X}$.

[24] Milgrom and Weber (1982) give the following comment on the value estimate: To represent a bidder's information by a single real-valued signal is to make two substantive assumptions. Not only must his signal be a sufficient statistic for all of the information he possesses concerning the value of the object to him, it must also adequately summarize his information concerning the signals received by the other bidders. It is in the light of these difficulties that we choose to view each signal as a value estimate.

[25] The main purpose with these additional signals is that they allow us to model the case where the value of the good has a random component even after all the signals $\mathbf{X}$ have been observed. An affiliation condition on $\boldsymbol{S}$ imposes some structure on the residual randomness.

[26] Note that our Assumption 2 is slightly more restrictive compared to Milgrom and Weber (1982). We assume that the function u is strictly increasing in x_i, whereas they assume that it is weakly increasing. Our stricter assumption is useful when proving Lemma 1 and its implications.

[27] Milgrom & Weber (1982) only assume that $E[V_i]$ is bounded for each i. We need a stronger assumption as we rule out non-monotonic symmetric equilibria.

We define the set $K_Z = \{k | \underline{Z} \leq k \leq \overline{Z}\}$ as the set of Z values that occur with positive probability.

We further assume that each signal X_i has possible values given by the interval $Q_X = (a_L, a_U)$ where $a_L \in \mathbb{R} \cup \{-\infty\}$ and $a_U \in \mathbb{R} \cup \{+\infty\}$. Similarly the signals $\mathbf{S}$ have possible values Q_S, which may be unbounded. We will want to consider only signals that occur with a positive density, so we assume that for every $\boldsymbol{S} \in Q_S$, and for each $x \in Q_X$, the density $f_k(\boldsymbol{S}, x, \mathbf{X}_{-1}) > 0$ for some $\mathbf{X}_{-1}$ and k.

Consider bidder 1 and let Y_k denote the kth highest signal of its competitors, and $\boldsymbol{Y}$ be a vector of those signals. Due to symmetry properties and Assumption 1, we can write the value of bidder 1 as follows:

$$V_1 = u(\boldsymbol{S}, X_1, \mathbf{Y}).$$

We also define Y_Z to be the Zth highest signal of the competitors, a random variables that is determined from the values taken by the random variables $\boldsymbol{X}_{-i}$ and Z. This variable plays a central role in our analysis, similar to Y_1 in Milgrom & Weber (1982), who assume that $Z = 1$ with certainty. We call Y_Z the signal of the marginal competitor. Note that Y_Z is an exogenous variable, it does not depend on bids.

Analogous to Milgrom & Weber (1982), we make the following standing assumptions for the affiliated-signals model.

Assumption 4: f_k is symmetric in its last n arguments.
Assumption 5: The variables $S_1, \ldots, S_m, X_1, \ldots, X_n, Y_Z$ are affiliated.

Affiliation is a strong version of positive correlation, and is formally defined below. Assumption 5 is the affiliation condition that we mentioned in the introduction. A difference to Milgrom & Weber (1982) is that we have added Y_Z to the list of affiliated signals. This puts structure on the uncertainty in Z. If Z is certain, as in Milgrom & Weber (1982), then Y_Z is automatically affiliated with the other signals. We discuss the affiliation condition further in Section 3.1. Assumption 1 and 4 ensure that the game is symmetric ex ante, before bidders observe any signals. A strategy for bidder i is a function mapping its value estimate X_i into a bid $b = b_i(X_i) \geq 0$.

In this section, we will study a uniform-price auction with single-unit demand. The highest rejected bid sets the clearing price, so it corresponds to a Vickrey auction. In case of ties, acceptance is determined randomly such that each bid at the clearing price has the same chance of being accepted. We will solve for a pure-strategy Bayesian Nash Equilibrium (BNE), where each bidder chooses an optimal bid conditional on the bidding strategies of the competitors and conditional on the information that it observes. Similar to Milgrom & Weber (1982) we will solve for symmetric equilibria for the affiliated-values model. Without loss of generality, we will focus on the bidding decision by bidder 1. To reduce issues with irregularities in the bid functions, we will work with a slight refinement of the Bayesian Nash Equilibrium (BNE). Even if a value occurs with measure zero, it makes sense for a bidder to bid rationally once the value has been realized. Hence, for our equilibria, we will require that each bidder maximizes its expected profit for every observed private signal. This is somewhat stronger than the standard definition of the BNE, which would allow players to act irrationally for a finite number of events that occur with measure zero. By making this definition we make some of the proofs less technical. The simplification only changes equilibrium outcomes for events that occur with measure zero.

Affiliation and related concepts are formally defined in Milgrom & Weber (1982). We repeat some of those definitions here. A subset L of $\mathbb{R}^k$ is a *sublattice* if u and v in L imply that the vectors $u \vee v$ and $u \wedge v$ are also in L (where $(u \vee v)_i = \max(u_i, v_i)$, $(u \wedge v)_i = \min(u_i, v_i)$, $i = 1, ..., k$). A subset A of $\mathbb{R}^k$ is *increasing* if its indicator function I_A is non-decreasing. In other words if $x \in A$ and $y_i \geq x_i$, $i = 1, ..., k$ then $y \in A$. Let $Z = (Z_1, ..., Z_k)$ be a random vector. $Z_1, ..., Z_k$ are *associated* if for all increasing sets A and B, $\Pr(A \cap B) \geq \Pr(A)\Pr(B)$, and $Z_1, ..., Z_k$ are *affiliated* if for all increasing sets A and B, and every sublattice L, $\Pr(A \cap B \mid L) \geq \Pr(A \mid L)P(B \mid L)$, i.e., if the variables are associated conditional on any sublattice. In the case where there are densities, Milgrom & Weber (1982) show that if variables $Z_1, ..., Z_k$ are affiliated for any vectors z and z' that are possible realisations of the Z_i, then

$$f(z \wedge z')f(z \vee z') \geq f(z)f(z') \tag{1}$$

where f is the joint density of the variables $Z_1, ..., Z_k$.

3.1 Affiliation of the marginal competitor's signal

One property that is important for our analysis is that Y_Z is affiliated with the other signals, as assumed in Assumption 5. Note that this is an exogenous assumption that only depends on properties of the joint probability density of signals and Z. Moreover, it follows from the argument made in Theorem 2 of Milgrom & Weber (1982) that this property follows if Z is certain and $S_1, \ldots, S_m, X_1, \ldots, X_n$ are affiliated. In Proposition 1 below we prove that this is also the case when Z is independent of the affiliated signals $S_1, \ldots, S_m, X_1, \ldots, X_n$. Hence, assuming that Y_Z is affiliated with the signals of the bidders and the auctioneer is only restrictive when Z is correlated with these signals.[28]

Proposition 1 *If $S_1, \ldots, S_m, X_1, \ldots, X_n$ are affiliated variables that are independent of Z, then $S_1, ..., S_m$,X_1, $Y_1, ...Y_{n-1}$, Y_Z are affiliated.*

4 Characterising symmetric equilibrium

Milgrom & Weber (1982) consider bid strategies that are increasing with respect to signals and solve for a single symmetric equilibrium in a second-price auction. In this subsection we take this a step further. We allow for multiple sold units and weaken assumptions on the bid functions, other than that the equilibrium is symmetric. In particular we will not restrict ourselves to continuous or strictly

[28] Note that all proofs are in the Appendix.

increasing bid functions. Rather we will show that these properties follow from the fact of a symmetric equilibria over affiliated signals, and assumptions made in Section 3. The only restriction we make is that the bid functions are piecewise continuous, where we assume that each piece is increasing, decreasing or constant. We say that a bid function $b(x)$ defined on Q_X is *regular* if there are a finite number of break points $a^{(1)} < a^{(2)} < ... < a^{(M+1)}$ where we take $a^{(1)} = a_L \in \mathbb{R}\cup\{-\infty\}$ and $a^{(M+1)} = a_U \in \mathbb{R}\cup\{+\infty\}$, and $b(x)$ is either (i) continuous and strictly increasing; or (ii) continuous and strictly decreasing; or (iii) constant; in each of the intervals $(a^{(\ell)}, a^{(\ell+1)})$ for $\ell = 1, 2, ..., M$. We assume that the value of the bid functions at the points $a^{(\ell)}$, $\ell = 1, 2, ..., M$, are defined by continuity either on the right or left. The value of the bid function outside of the set Q_X is immaterial. We say that an equilibrium is regular if the bid functions used are regular.

In the next subsection we will show existence of an equilibrium that is symmetric and regular. If Z is independent of the signals of the bidders, we can also show that there is a single equilibrium that is symmetric and regular. This is a long argument that is established in stages. In Section 4.2, we will verify that the publicity effect holds for the symmetric equilibrium.

4.1 Existence and uniquenessof symmetric equilibrium

Lemma 1 below is a slightly stronger version than Theorem 5 in Milgrom & Weber (1982) in the sense that a bidder's expected value is strictly increasing with respect to its own signal. We are able to prove this result as our Assumption 2 is slightly stronger than in Milgrom & Weber (1982).

Lemma 1 *For any sublattice L, the function $\mathbb{E}[\, V_1 |\, X_1 = x, \widetilde{\mathbf{X}} \in L]$ is strictly increasing in x for $x \in Q_X$.*

We focus on firm 1 and write W_Z for the Zth highest bid amongst the other bidders (whereas Y_Z is the Zth highest signal). In equilibrium, the bid of firm 1 depends on its valuation of the good. In its turn the valuation depends on its own signal, but also on competitors' signals, which are not directly observable. A marginal change in the bid only matters when the bid is on the margin of being accepted, i.e. when the bid b equals W_Z. Hence, this is the event that bidder 1 will condition on when it optimizes its bid. It follows that $\mathbb{E}[V_1 |\, X_1 = x, W_Z = b]$ is going to be a key quantity in our analysis. Given a set of bid functions for the other bidders, we define:

$$v_W (x, b) = \mathbb{E}\left[V_1 |\, X_1 = x, W_Z = b\right],$$

so this is the expected value for bidder 1 given a signal x and given that the Zth highest bid among the competitors is b. We begin by establishing that $\mathbb{E}[V_1 |\, X_1 = x, W_Z = b, Z = k]$, for $k \in K_Z$ is increasing in x. Here we consider a set of equilibrium bids, but they only appear in the expected value through the conditioning on $W_Z = b$, so this expected value does not depend on the bid of player 1. Though we can establish that the expected value is increasing in X_1

when Z is given, the corresponding result for $v_W(x, b)$ may not hold when Z is correlated with the signals.

Lemma 2 *In a symmetric regular equilibrium with bid functions b^* defined on Q_X, $\mathbb{E}[V_1 | X_1 = x, W_Z = b_0, Z = k]$ is strictly increasing in x for each $b_0 \in b^*(Q_X)$ and $k \in K_Z$. If Z is independent then $v_W(x, b_0)$ is strictly increasing in x for $x \in Q_X$.*

Lemma 3 *If Z is independent, then in a symmetric regular equilibrium with bid functions b^* defined on Q_X, there cannot be an interval in which b^* is constant.*

Suppose that b^* is a regular bid function, with break points $a^{(1)}, a^{(2)}, ...a^{(M+1)}$. Then we set $G(b^*)$ to be the set of bid values at the ends of the intervals $(a^{(j)}, a^{(j+1)})$, defined by the bid $b^*(x)$, so $G = \{y : \lim_{\delta \searrow 0} b^*(a^{(k)} + \delta) = y$ or $\lim_{\delta \searrow 0} b^*(a^{(k)} - \delta) = y$, for some $k = 1, 2, ..., M + 1\}$.

Lemma 4 *If Z is independent, then in a symmetric regular equilibrium with bid functions b^*, $v_W(x, b)$ is continuous in b for $x \in Q_X$ provided b is not in $G(b^*)$.*

The decision problem of bidder 1 is to choose a bid b that maximizes its expected payoff, assuming that for $i \neq 1$ bidder i uses the bid function b_i^*. In the case that b^* is a regular equilibrium, then Lemma 3 implies that there are no segments of constant value (no accumulation of bids at any price) and this ensures that the probability of the event $W_Z = b$ is zero. Thus we have the expected payoff given by:

$$\max_b \quad \mathbb{E}\left[(V_1 - W_Z) 1_{W_Z < b} \middle| X_1 \right],$$

where $1_{W_Z < b}$ is an indicator function. Moreover this expression will not change its value if $1_{W_Z < b}$ is replaced by $1_{W_Z \leq b}$.

Lemma 5 *In a symmetric regular equilibrium with bid functions b^*, for any $x \in Q_X$ in the interior of an interval where b^* is continuous, we have $b^*(x) = v_W(x, b^*(x))$.*

Lemma 6 *If Z is independent, then in a symmetric regular equilibrium the bid function b^* is strictly increasing for $x \in Q_X$.*

At this point we have established that restricting our attention to regular symmetric equilibria implies that the optimal bid consists of segments each of which is continuous and strictly increasing. If there is a jump then it can only be a jump upwards. The final step is to rule out jumps in the bid function.

Lemma 7 *In a symmetric regular equilibrium with independent Z, the bid function b^* is continuous for $x \in Q_X$.*

The next result establishes that there is a unique symmetric (regular) equilibrium.

Lemma 8 *For independent Z, there is only one symmetric regular equilibrium.*

We define

$$v(x,y) = \mathbb{E}\left[V_1 | X_1 = x, Y_Z = y\right]$$

and note that for strictly increasing symmetric bid functions

$$\begin{aligned} v_W(x, b(y)) &= \mathbb{E}\left[V_1 | X_1 = x, b(y) = W_Z\right] = \mathbb{E}\left[V_1 | X_1 = x, Y_Z = y\right] \\ &= v(x,y). \end{aligned} \tag{2}$$

We can now combine all these Lemmas to give the following result.

Proposition 2 *If Z is independent then in a symmetric regular equilibrium each player uses a bid function b^*, which is continuous, strictly increasing and satisfies $b^*(x) = v(x,x)$, and there can be at most one such equilibrium.*

We can also make deductions about the existence of an equilibrium without the assumption on Z being independent. But in this case we cannot rule out the possibility of other less well-behaved equilibria.

Proposition 3 *There is a symmetric equilibrium with a continuous strictly increasing bid function*

$$b^*(x) = v(x,x)$$

and this is the only symmetric equilibrium with a continuous and strictly increasing bid function.

Hence, as long as Y_Z is affiliated and Assumption 5 is satisfied, then the existence of monotonic symmetric equilibrium bids can always be ensured. We know from Proposition 1 that Assumption 5 is satisfied when Z is independent of the signals. It is natural to look for other conditions for which Y_Z is affiliated. This requires Y_Z to increase when the signals increase, so it is natural to look at cases where $-Z$ is affiliated with the other signals. This would imply that increasing signals reduces the number of bids accepted which tends to increase Y_Z. We give an example below to show that $-Z$ being affiliated with the signals is not sufficient to ensure that Y_Z is affiliated.

Example 1 *There are three bidders, each receives a private signal $X_i \in (0,2)$. The common value of the object (to any of the bidders) is determined from the signals and is given by $V = \sum X_i$. Signals are uniformly distributed on $(0,2)$ and are independent. The number of items auctioned, Z, is also determined by the signals. If two or three of the signals are in the range $(0,1)$ then $Z = 2$, and* *otherwise* $Z = 1$. *Thus the set in* $(0,2)^3$, *on which* $Z = 1$, *is increasing and this is enough to show* $-Z$ *is affiliated with the signals. To prove this we consider arbitrary increasing sets* A *and* B *in* $[0,2]^3 \times \{-1,-2\}$ *and an arbitrary sublattice* L *in* $[0,2]^3 \times \{-1,-2\}$. *Then we define* $Z(X_1, X_2, X_3)$ *as the value of* Z *given signals* (X_1, X_2, X_3) *and map the sets* A, B *and* L *into sets* $\widetilde{A}$, $\widetilde{B}$ *and* $\widetilde{L}$ *in* $[0,2]^3$ *in such a way that* $(X_1, X_2, X_3, -Z(X_1, X_2, X_3)) \in A$ *exactly when* $(X_1, X_2, X_3) \in \widetilde{A}$, *and similarly with* $\widetilde{B}$ *and* $\widetilde{L}$. *We can then show that* A *and* B *increasing implies that* $\widetilde{A}$ *and* $\widetilde{B}$ *are also increasing, using the property that the set where* $Z(X_1, X_2, X_3) = 1$ *is increasing*[29]. *Next, we note that* $-Z$ *being an increasing function implies that* $\widetilde{L}$ *is a sublattice.*[30] *The property we need for* $-Z$ *affiliated can now be established.*

$$\begin{aligned}
\Pr((X_1, X_2, X_3, -Z) &\in A \cap B|\, L) = \Pr((X_1, X_2, X_3) \in \widetilde{A} \cap \widetilde{B} \Big| \widetilde{L}) \\
&\geq \Pr((X_1, X_2, X_3) \in \widetilde{A} \Big| \widetilde{L}) \Pr((X_1, X_2, X_3) \in \widetilde{B} \Big| \widetilde{L}) \\
&= \Pr((X_1, X_2, X_3, -Z) \in A|\, L) \Pr((X_1, X_2, X_3, -Z) \in B|\, L).
\end{aligned}$$

We evaluate $v(x,x) = \mathbb{E}[V \mid X_1 = x, Y_Z = x]$ *at* $x = 1 + \delta, \delta > 0$ *and small. Assume without loss of generality that* $X_2 < X_3$. *If both* $X_2, X_3 \in (0,1)$ *then* $Z = 2$, *and* $Y_Z < X_1$, *so* $X_1 = Y_Z = x$ *can be ruled out. Hence,* $Z = 1$ *in which case* $Y_Z = x$ *implies* $X_3 = x$ *and* $X_2 < x$. *Now, conditioning on* $X_1 = Y_Z = x$ *gives an expected value where* X_2 *is uniform on* $(0, 1+\delta)$, *so*

$$v(x,x) = 2(1+\delta) + (1+\delta)/2 = \frac{5}{2} + \frac{5}{2}\delta$$

and approaches $5/2$ *as* x *approaches* 1 *from above. Next we evaluate* $v(x,x) = \mathbb{E}[V \mid X_1 = x, Y_Z = x]$ *at* $x = 1-\delta, \delta > 0$ *and small. In the cases where* $Z = 1$, *we have* X_2 *and* $X_3 \in (1,2)$ *so we do not have* $Y_Z = X_3 = X_1$. *So consider cases where* $Z = 2$, *and hence* $X_2 \in (0,1)$ *and* $X_3 \in (X_2, 2)$. *Thus* $Y_Z = X_1 = 1 - \delta$ *implies* $X_3 \in (1-\delta, 2)$ *and is equally likely to take any value in this range.* $(1-\delta, 2)$. *So*

$$v(x,x) = 2(1-\delta) + \frac{3-\delta}{2} = \frac{7}{2} - \frac{5}{2}\delta$$

and approaches $7/2$ *as* x *approaches* 1 *from below. Hence the value of* $v(x,x)$ *jumps down at* $x = 1$. *Thus* Y_Z *is not affiliated, since if it was we get a contradiction from Proposition 3.*

[29] We can define $\widetilde{A}_i = \{(X_1, X_2, X_3) \mid (X_1, X_2, X_3, i) \in A, -Z(X_1, X_2, X_3) = i\}$ for $i \in \{-1,-2\}$, so that $\widetilde{A} = \widetilde{A}_{-1} \cup \widetilde{A}_{-2}$. Then consider any $\mathbf{X} \in \widetilde{A}$ and suppose that $\mathbf{X}' \geq \mathbf{X}$ component-wise. If $\mathbf{X} \in \widetilde{A}_{-1}$ so $(\mathbf{X}, -1) \in A$ and $Z(\mathbf{X}) = 1$, then $(\mathbf{X}', -1) \in A$, and $Z(X') = 1$ because of the assumed properties of the set on which $Z = 1$, and thus $\mathbf{X}' \in \widetilde{A}_{-1}$. On the other hand if $\mathbf{X} \in \widetilde{A}_{-2}$, we have $(\mathbf{X}, -2) \in A$ and $Z(\mathbf{X}) = 2$. Since A is an increasing set both $(\mathbf{X}', -2) \in A$ and $(\mathbf{X}', -1) \in A$. Thus if $Z(\mathbf{X}') = 1$ then $\mathbf{X}' \in \widetilde{A}_{-1}$, and if $Z(\mathbf{X}') = 2$ then $\mathbf{X}' \in \widetilde{A}_{-2}$. Thus in all cases $\mathbf{X}' \in \widetilde{A}$. Hence $\widetilde{A}$ is an increasing set.

[30] Since $-Z$ is an increasing function, $\min(-Z(X_1, X_2, X_3), -Z(X_1', X_2', X_3')) = -Z((X_1, X_2, X_3) \wedge (X_1', X_2', X_3'))$ and $\max(-Z(X_1, X_2, X_3), -Z(X_1', X_2', X_3')) = -Z((X_1, X_2, X_3) \vee (X_1', X_2', X_3'))$. Thus when both $(X_1, X_2, X_3, -Z(X_1, X_2, X_3))$ and $(X_1', X_2', X_3', -Z(X_1', X_2', X_3'))$ are in L we can deduce that $(X_1, X_2, X_3) \wedge (X_1', X_2', X_3')$ and $(X_1, X_2, X_3) \vee (X_1', X_2', X_3')$ are in $\widetilde{L}$.

4.2 Publicity effect

Now, assume that the seller, the auctioneer, has a signal X_0 that it might want to disclose to all bidders. This signal is affiliated with the other signals and with Y_Z. It can be taken to be one of the **S** signals in the definitions we had previously. Similar to Milgrom & Weber (1982), we use the superscript $\mathbb{N}$ for markets where X_0 is not disclosed and the superscript $\mathbb{I}$ when the auctioneer's information is made public. If X_0 is disclosed to all bidders, then private signals will be drawn from a new probability density that is conditional on X_0. It can be shown that the remaining signals (i.e. signals except for X_0) are still affiliated (Milgrom & Weber, 1982). Thus results in Section 3.1 and 4.1 will also hold for the new conditional distribution. For example, there will be a symmetric BNE where each bidder $i \in \{1, \ldots, n\}$ has a strictly increasing bid function $\hat{b}^*(x; x_0) = \hat{v}(x, x; x_0)$, where

$$\hat{v}(x, y; x_0) = \mathbb{E}\left[V_1 | X_1 = x, Y_Z = y, X_0 = x_0\right]. \tag{3}$$

It follows from Theorem 5 in Milgrom & Weber (1982) that $\hat{v}$ is non-decreasing in its arguments. Let

$$\begin{aligned} P^{\mathbb{I}}(X_1) &= \mathbb{E}\left[\hat{v}(Y_Z, Y_Z; X_0) | X_1 > Y_Z\right] \\ P^{\mathbb{N}}(X_1) &= \mathbb{E}\left[v(Y_Z, Y_Z) | X_1 > Y_Z\right] \end{aligned}$$

be the expected payment from bidder 1 when it observes X_1 and gets the offer accepted.

Lemma 9 *For the uniform-price auction we have $P^{\mathbb{I}}(x) \geq P^{\mathbb{N}}(x)$, for strictly increasing symmetric equilibria. We have $P^{\mathbb{I}}(x) = P^{\mathbb{N}}(x)$ if X_0 is independent of $X_1, \ldots, X_n$ and Z.*

The ranking of revenues is more relevant for an auctioneer selling multiple items. Hence, we define the following expected payment to the auctioneer from bidder 1, or any bidder, observing the signal x.[31]

$$R(x) = \mathbb{E}\left[P(x) 1_{Y_Z < x} | X_1 = x\right] = P(x) \Pr(Y_Z < x | X_1 = x), \tag{4}$$

for both $\mathbb{I}$ and $\mathbb{N}$ cases and we have dropped the superscript. The probability $\Pr(Y_Z < x | x)$ is the same irrespective of whether X_0 is disclosed or not. Hence, it follows from Lemma 9 that $R^{\mathbb{I}}(x) \geq R^{\mathbb{N}}(x)$, and we can conclude that:

Corollary 1 *For strictly increasing symmetric equilibria in the uniform-price auction, we have $\mathbb{E}\left[P^{\mathbb{I}}(x)\right] \geq \mathbb{E}\left[P^{\mathbb{N}}(x)\right]$ and $\mathbb{E}\left[R^{\mathbb{I}}(x)\right] \geq \mathbb{E}\left[R^{\mathbb{N}}(x)\right]$. Hence, both the expected price and the expected revenue of the auctioneer weakly increases when it discloses its signal X_0. We have $\mathbb{E}\left[P^{\mathbb{I}}(x)\right] = \mathbb{E}\left[P^{\mathbb{N}}(x)\right]$ and $\mathbb{E}\left[R^{\mathbb{I}}(x)\right] = \mathbb{E}\left[R^{\mathbb{N}}(x)\right]$ if X_0 is independent of $X_1, \ldots, X_n$ and Z.*

[31] Note that this expression is also valid when X_0 is disclosed.

$$\begin{aligned} R^{\mathbb{I}}(x) &= E\left[\hat{v}(Y_z, Y_z, X_0) 1_{Y_z < x} | X_1 = x\right] = E\left[\hat{v}(Y_z, Y_z, X_0) | X_1 = x, Y_z < x\right] \Pr(Y_z < x | X_1 = x) \\ &= P^{\mathbb{I}}(x) \Pr(Y_z < x | X_1 = x) \end{aligned}$$

Theorem 8 in Milgrom & Weber (1982) uses the ranking of expected selling prices to prove the publicity effect for single-object, second-price auctions. We generalize this result to a uniform-price, multi-unit auction with single-unit demand and uncertain supply. But more importantly we show that, even if signals of bidders and the auctioneer's supply are correlated, one gets the same ranking for expected revenues.

One implication of our result is that even if X_0 would include information that is relevant for the valuation of the good, the auctioneer would not gain anything (in expectation) from disclosing X_0, unless the information can also be used by a bidder to predict competitors' signals or the number of traded objects. Pycia & Woodward (2021) provides an example of the latter. They consider a uniform-price auction with divisible-goods and symmetric information, where disclosing the auctioneer's supply increases the revenue of the auctioneer.

Similar to Milgrom & Weber (1982) an auctioneer could consider a more complicated policy, to disclose $X_0' = r(X_0, \vartheta)$, so that the disclosed signal X_0' is a function of X_0 and a random variable ϑ, which is independent of all signals. But this would worsen the revenue of the auctioneer.

Proposition 4 *For the uniform-price auction, disclosing the signal X_0 gives a weakly higher expected price and weakly higher expected revenue of the auctioneer compared to any reporting policy $r(X_0, \vartheta)$.*

5 The linkage principle

The linkage principle was used by Milgrom & Weber (1982) to rank first- and second-price auctions and to prove the publicity effect in first-price auctions. In this section, we will extend the linkage principle so that it can be used to rank uniform-price and pay-as-bid auctions for single-unit demand. Holmberg & Wolak (2018) made extensions of the linkage principle, so that it could rank expected revenues in their divisible-good model with two bidders. Here we make an additional extension, so that the linkage principle can be used for multiple bidders and when the number of traded units is correlated with the signals $\boldsymbol{X}$.

Consider a symmetric equilibrium, where each bidder submits a bid $b(x)$ when observing the private signal x. Assume that the competitors follow this equilibrium strategy, and that $b(x)$ is strictly monotonic. However, we allow the considered bidder (bidder 1) to deviate and act as if observing a signal $\widetilde{x}$, i.e. it can make an offer $b(\widetilde{x})$, although it actually observes the signal x.

Conditional on having the bid accepted, let $J(\widetilde{x}, x) \geq 0$ be the expected payment when the bidder observes $x \in Q_X = (a_L, a_U)$ and bids as if observing $\widetilde{x} \in Q_X$. Recall that Z is the number of sold units and that Y_Z is the Zth highest signal among the competitors of bidder 1. The equilibrium is symmetric and strictly monotonic, so bidder 1 needs to act as if having a higher signal than Y_Z to have its bid accepted. Hence, the probability that its bid is accepted is given by $\Pr(Y_Z \leq \widetilde{x} | x)$, where $\Pr(Y_Z \leq \widetilde{x} | x)$ is non-decreasing with respect to $\widetilde{x}$. We also introduce the expected payment to the auctioneer from bidder 1 and bidder 1's expected utility of the good :

$$
\begin{aligned}
K(\widetilde{x}, x) &= J(\widetilde{x}, x) \Pr(Y_Z \leq \widetilde{x} | x) && (5)\\
U(\widetilde{x}, x) &= \mathbb{E}[V_1 | Y_Z \leq \widetilde{x}; x] \Pr(Y_Z \leq \widetilde{x} | x). && (6)
\end{aligned}
$$

For the moment, we assume that $J(\widetilde{x}, x)$, $U(\widetilde{x}, x)$ and $K(\widetilde{x}, x)$ are bounded and differentiable functions with respect to both arguments, and we use subscripts $\widetilde{x}$ and x for these partial derivatives. Later when we apply the linkage principle in the next section, we will add an extra differentiability assumption on the value function and distribution of signals to make sure that this is the case. From (5) we have that

$$
\lim_{\widetilde{x} \searrow a_L} K(\widetilde{x}, x) = 0, \text{ for } x \in Q_X. \tag{7}
$$

Our next lemma is a technical result which shows what the ranking of J_x implies for the ranking of J. Related results have been proven by Milgrom & Weber (1982) for single-object auctions. The argument is slightly more involved for multi-unit auctions. We compare two auction designs indicated by superscripts I and II.

Lemma 10 *For two auction designs we have* $J^{\mathbb{I}}(x, x) \geq J^{\mathbb{II}}(x, x)$ *for* $x \in Q_X$ *if* $J_x^{\mathbb{I}}(x, x) \geq J_x^{\mathbb{II}}(x, x)$ *for* $x \in Q_X$ *and* $\lim_{x \searrow a_L} J^{\mathbb{I}}(x, x) \geq \lim_{x \searrow a_L} J^{\mathbb{II}}(x, x)$.

We are now ready to fully generalise the linkage principle for single-object auctions to our multi-unit setting. An auction design that increases the linkage between a bidder's private signal and its expected payment to the auctioneer conditional on acceptance is beneficial for the auctioneer. Its expected revenue increases, and also the expected payment that it receives. Lemma 10 has established conditions under which the expected price is increased under a change in auction design, we now show that the same conditions will ensure that the expected revenue of the auctioneer increases, even if signals and Z would be correlated.

Lemma 11 *If for two auction designs* $\lim_{x \searrow a_L} J^{\mathbb{I}}(x, x) \geq \lim_{x \searrow a_L} J^{\mathbb{II}}(x, x)$ *and* $J_x^{\mathbb{I}}(x, x) \geq J_x^{\mathbb{II}}(x, x)$ *for* $x \in Q_X$, *then* $K^{\mathbb{I}}(x, x) \geq K^{\mathbb{II}}(x, x)$ *for* $x \in Q_X$.

6 Pay-as-bid auction

In this section, we will consider a well-behaved symmetric equilibrium in a pay-as-bid auction and make comparisons with the uniform-price auction. Our results for the pay-as-bid auction generalizes corresponding results for the first-price auction in Milgrom & Weber (1982). The following simplifying assumptions are used to prove the lemma below, which simplifies the characterization of the pay-as-bid auction and the application of the linkage principle:

Assumption 6: $f_k(\boldsymbol{S}, \boldsymbol{X})$ and $V_1 = u(\boldsymbol{S}, X_1, \boldsymbol{X}_{-1}) = u(\boldsymbol{S}, X_1, \boldsymbol{Y})$ are both differentiable with respect to $\boldsymbol{X}$ and $f_k(\boldsymbol{S}, \boldsymbol{X}) > 0$ if $X_i \in Q_X$ for $i = 1, \ldots, n$ and $k \in K_Z$.

Now, we are ready to study the pay-as-bid auction. Assume that each competitor $i \neq 1$ observes a signal x_i and bids in accordance with the strategy $b^*(x_i)$, which is a strictly increasing (and invertible) function. The expected payoff for bidder 1 observing signal x and bidding b in a pay-as-bid auction is then:

$$\begin{aligned}
\Pi(b,x) &= \mathbb{E}\left[(V_1 - b)\, 1_{W_Z<b} \middle| X_1 = x\right] \\
&= \int_{a_L}^{b^{*-1}(b)} (v(x,\alpha) - b)\, f_{Y_Z}(\alpha | x)\, d\alpha,
\end{aligned} \tag{8}$$

where $f_{Y_Z}(\cdot| x)$ is the probability density of Y_Z (and $F_{Y_Z}(\cdot| x)$ is the corresponding cdf) conditional on bidder 1 observing x.

Proposition 5 *In a pay-as-bid auction, there is a symmetric equilibrium with the bid function*

$$\begin{aligned}
b^*(x) &= v(x,x) - \int_{a_L}^{x} \exp\left(-\int_{\alpha}^{x} \frac{f_{Y_Z}(s| s)}{F_{Y_Z}(s| s)} ds\right) dt(\alpha) \\
t(\alpha) &= v(\alpha,\alpha),
\end{aligned} \tag{9}$$

which is continuous and strictly increasing for $x \in Q_X$.

For the special case where Z is certain and equal to 1, the equilibrium is the same as the equilibrium that Milgrom & Weber (1982) find for first-price auctions. Similarities between pay-as-bid and first-price auctions have also been pointed out by Wittwer (2018).

6.1 Publicity effect

Now, assume that the seller has a signal X_0 that is affiliated with the other signals, including Y_Z, and that it might want to disclose to all bidders. It could for example be one of the signals in $\mathbf{S}$. If X_0 is disclosed to all bidders, then private signals will be drawn from a new probability density that is conditional on X_0. The new signals are still affiliated (Milgrom & Weber, 1982). Thus the results that we proved above will also hold for a pay-as-bid auction with the new conditional distribution. As before, we use the superscript $\mathbb{N}$ for markets where X_0 is not disclosed and the superscript $\mathbb{I}$ when the auctioneer's information is made public.

Proposition 6 *In a pay-as-bid auction, which satisfies Assumption 6, the expected bid price and the expected revenue of the auctioneer weakly increases when the auctioneer discloses its signal X_0. There is no effect if X_0 is independent of $X_1, \ldots, X_n$ and Z.*

Hence, similar to the uniform-price auction, disclosing X_0 only increases the expected revenue of the auctioneer if X_0 is somewhat informative of competitors' signals or of the auctioneer's supply. Using a similar argument as in the proof of Proposition 4, it can be shown that:

Proposition 7 *For the pay-as-bid auction, disclosing the signal X_0 gives a weakly higher expected price and weakly higher expected revenue of the auctioneer compared to any reporting policy $r(X_0, \vartheta)$.*

6.2 Ranking of auctions

The linkage-principle argument can also be used to rank uniform-price and pay-as-bid auctions. The result below extends Milgrom & Weber's (1982) ranking of first- and second-price auctions, and generalizes the ranking result in Weber (1983) for an auctioneer with a fixed supply.

Proposition 8 *Under Assumption 6, the expected price for an accepted bid in the uniform-price auction and the expected revenue of the auctioneer of such an auction are both at least as large as in the pay-as-bid auction. The expected price and expected revenue are the same in the two auctions if $X_1, \ldots, X_n$ and Z are all independent.*

7 Private-value model

Previous sections have studied symmetric BNE for auctions where bidders are symmetric ex ante. In Section 4.1 we showed that such equilibria are well behaved, especially if signals of bidders and Z are independent. But from practice we know that uniform-price auctions can have prices at the collusive level. In this section, we will solve for all pure-strategy equilibria in the uniform-price auction, including asymmetric equilibria. We will show that there are asymmetric equilibria with prices at the collusive level and how one can get rid of such outcomes. Our model will be less restrictive in some aspects than in previous sections. We will allow bidders to be asymmetric ex ante, we will allow bidders to be imperfectly informed of the joint probability distribution of signals and Z, and we will drop Assumptions 1-6. But in order to achieve this we need to introduce restrictive assumptions on the function $u_i(\boldsymbol{S}, \boldsymbol{X})$.

7.1 Assumptions for the private-value model

The signal of a bidder is allowed to be correlated with signals of competitors, but we will assume that the value of a bidder, conditional on its signal, is independent of competitor's signals. This corresponds to the private-value case. It does not matter whether a bidder directly observes its valuation of the object or works with an expected valuation $u_i(\boldsymbol{S}, X_i)$, as long as the latter is independent of competitors' signals. This also means that we can drop the vector of unobserved signals $\boldsymbol{S}$ from the analysis. As before we assume that the value function is increasing and continuous. Hence, signals and values are equivalent and we assume that each bidder observes a signal equal to its private value.

We suppose that the values for firm i are in the open interval $(\underline{V}_i, \overline{V}_i)$ with a positive probability density throughout this range, without any single points where there is a positive probability mass. These open intervals can vary between firms. We allow Z and signals to be drawn from a general probability distribution. Signals could for example be positively correlated or negatively correlated, so there is no assumption of affiliated signals in this section.

Moreover, we assume that the range of possible values and positivity of densities remain unchanged no matter the value of Z, or what signals are observed by competitors, and that these ranges are common knowledge, even though the joint distribution of signals may not be common knowledge. We restrict bids to regular bids, as for the affiliated case.

The set of equilibria depend on the value ranges of the bidders, but our results do not depend on the joint distribution of Z and the signals. For the set of equilibria it does not matter whether the bidders know this joint distribution. The bidders could even have different beliefs about the joint distribution.

7.2 The reflected auction

Before we start to analyse the private-value model, we will prove a transformation property that simplifies our analysis. Suppose that we have an equilibrium for auction A with Z items auctioned in the private-value case. First, consider a transformed auction (call it auction B) in which all values are changed through a positive affine map. So where a firm had a previous value of v it has a new value of $\alpha v + \beta$, for some constants $\alpha > 0$ and β (with the same constants for each firm). Suppose that we have an equilibrium amongst bids in auction A, then we can transform this into a set of bids in auction B, with a bid of y in auction A translated into a bid of $\alpha y + \beta$. If all players translate their bids (and the auctioneer transforms any price cap or floor) in this way, then the probability of a bid being accepted remains the same in auction B as it was in auction A, with the clearing price transformed in the same way and expected profits multiplied by α. If a bid is optimal in auction A the transformed bid remains optimal in auction B, and form an equilibrium in this auction. Hence, positive affine transformations of payoffs in games do not influence the set of equilibria, which is a well-known result.

Now consider the case where α is negative, then there is a substantial change because the ordering between the values of different firms is reversed. Given results for positive affine transformations, it is sufficient to analyse the case with $\alpha = -1$ and $\beta = 0$ which we call the reflected auction. We will also change the values of Z in the reflected auction. For any auction A defined by the joint distribution of values for n firms and the auction quantity Z, we can define

Definition 1 *A reflected auction B is obtained by replacing each realisation of values $(v_1, v_2, ..., v_n)$ and quantity Z in Auction A, by the set of values $(-v_1, -v_2, ..., -v_n)$ and quantity $n - Z$.*

Note that negative bids and values are not an issue. Without changing the set of equilibria, we could add a constant to give positive values and bids in the reflected auction.

Lemma 12 *Suppose that firm i, $i = 1, 2, ...n$, makes a bid given by the function $b_i(x)$ for value x. This is an equilibrium in auction A if and only if there is an equilibrium in the reflected auction B in which firm i with value x, makes the bid $-b_i(-x)$, $i = 1, 2, ...n$.*

In essence this result shows that auctions in which all values are negated, and where the supply Z is replaced by $n - Z$ have the same equilibrium as before, we simply negate all the previous bids. If it was optimal to bid at value in the first auction it will be optimal to bid at value in the reflected auction. If a bid was accepted in the first auction it will be rejected in the reflected auction for the corresponding realisation, and vice versa. The equivalence of sales and procurement auctions is well-known, and at first glance this may seem a similar result, but in fact it is not directly related to this equivalence.

7.3 Partial high-low equilibria in the single-object auction and its reflection

Before proceeding with our analysis, we will also discuss Proposition 1 in Blume & Heidhues (2004) and its reflected version, as these results are helpful when interpreting our results.[32]

Proposition 1 in Blume & Heidhues (2004): *Consider the second-price sealed-bid auction with independent private values and $N \geq 3$ bidders. Suppose the distributions F_i, $i = 1, \ldots, N$, of valuations have positive densities f_i on the common support $[0, v_h]$: A strategy profile is a Nash equilibrium if it satisfies: There is a $\hat{b}$ such that:*

1. *any bidder with valuation $v > \hat{b}$ bids his valuation,*
2. *if $\hat{b} < v_h$, then there is one bidder who bids at $\hat{b}$ whenever his valuation v satisfies $v < \hat{b}$, and if $\hat{b} \geq v_h$ then there is one bidder who bids at or above $\hat{b}$ for any valuation v,*
3. *all other bidders bid 0 whenever their valuation v is in $[0, \hat{b})$.*

In summary, for low values $[0, \hat{b}) \subset [0, v_h]$, a single bidder bids high (at $\hat{b}$) and the rest low (at 0). The high bid is accepted and the low bids set the price, if all bidders have values below the threshold $\hat{b}$. All bid their value above $\hat{b}$. We refer to this as a high-low equilibrium at the bottom.[33] We refer to the degenerate case

[32] As the result is proven in Blume & Heidhues (2004), we present the result for their setting. But the result also holds for our setting which allows for correlated private values.

[33] Blume et al. (2009) refer to the equilibrium as the first class of equilibria.

where $\hat{b} = v_h$ as a high-low equilibrium.[34] In this case, one bid is high and the others low for the whole range of values. The other degenerate case where $\hat{b} = 0$ implies that all bid their value for the whole range of values. This corresponds to the well-behaved equilibrium. Blume & Heidhues (2004) show that other equilibria can be ruled out for their setting. Applying Lemma 12 to the proposition above immediately gives us a reflected version of the result in Blume & Heidhues (2004). As far as we know equilibria in the reflected auction, where there is one more bidder than the number of traded objects has not been studied in the previous literature, at least not in detail.

Corollary 2 *Consider the uniform-price auction with $N-1$ objects and $N \geq 3$ bidders that have single-unit demand and independent private values. Suppose the distributions F_i, $i = 1, \ldots, N$, of valuations have positive densities f_i on the common support $[0, v_h]$: A strategy profile is a Nash equilibrium if it satisfies: There is a $\hat{b}$ such that:*

1. *any bidder with valuation $v < \hat{b}$ bids his valuation,*
2. *if $\hat{b} > 0$, then there is one bidder who bids at $\hat{b}$ whenever his valuation v satisfies $v > \hat{b}$, and if $\hat{b} \leq 0$ then there is one bidder who bids at or below* 0 *for any valuation v,*
3. *all other bidders bid v_h whenever their valuation v is in $(\hat{b}, v_h]$.*

Again all other equilibria can be ruled out for the setting in Corollary 2. The high-low equilibrium at the bottom in the single-object auction becomes a high-low equilibrium at the top in the reflected auction. A single bidder bids low (at $\hat{b}$) and the others high (at v_h) for high values $(\hat{b}, v_h] \subset [0, v_h]$. The low bid sets the price and the high bids are accepted, if all bidders have values above $\hat{b}$. All bid their value below $\hat{b}$. The degenerate cases, the well-behaved equilibrium and the high-low equilibrium, remain in the reflected auction. The difference is that a single bidder bid low and the rest high in the high-low equilibrium.

One might wonder why a high-low equilibrium at the bottom does not exist in the reflected auction. The reason is that two or more bidders can only bid at the same price with a positive probability if those bids are either accepted with certainty or rejected with certainty. Otherwise one of those bidders would find it profitable to deviate and bid slightly higher. Hence, if, with a positive probability, there are bids at $\hat{b}$ in the middle of the range of equilibrium bids, then those bids must come from the same bidder. Thus, the high-low equilibrium at the bottom can only occur in a single-object auction, where one bid is always accepted, and the high-low equilibrium at the top can only occur in the reflected auction, where

[34] Blume et al. (2009) refer to this as the second-class of equilibria.

one bid is always rejected. At least this is the case if all bidders have the same range of values and there are at least three bidders.[35]

Below, we will simplify the analysis by choosing the number of objects such that it is strictly larger than one and such that at least two bids are always rejected. We will show that this assumption is sufficient to get rid of partial high-low equilibria, if bidders have the same value ranges. High-low equilibria exist as long as the auctioneer's supply is certain, but (as will be shown) they will disappear if the supply is uncertain. We will also consider the general case where bidders are allowed to have asymmetric value ranges. In this case, partial high-low equilibria can occur at the edges of the range of values, which are outside the value range of some bidders. In extensions, we will discuss how price floors, price caps and price-sensitive supply can be used to knock out high-low equilibria and partial high-low equilibria.

7.4 Equilibria in uniform-price auctions

Before proceeding with the analysis, we will introduce some new notation and some additional restrictions on the number of bidders and the value ranges. The following notation will be useful in our analysis. Choose a set Ω_U having size $\underline{Z}$ and containing the firms with the highest values of $\overline{V}_i$. Note that the set Ω_U may not be uniquely defined when there are common values of $\overline{V}_i$. We identify V_U as $\max\{\overline{V}_i : i \notin \Omega_U\}$ so that $\overline{V}_i \geq V_U$ for $i \in \Omega_U$ and $\overline{V}_i \leq V_U$ for $i \notin \Omega_U$. For a well-behaved equilibrium, where each firm bids its value, V_U would be the highest realized clearing price. We will show that any bid above V_U will be accepted with a positive probability. We define $\Omega'_U = \Omega_U \cup \{j_U\}$ where $j_U \notin \Omega_U$ is a firm with $\overline{V}_{j_U} = V_U$. We define $V'_U = \max\{\overline{V}_i : i \notin \Omega'_U\}$. Thus $V'_U \leq V_U$ and they may be the same (in case some firms have the same values of $\overline{V}_i$). We will show that any bid below V'_U will be rejected with a positive probability.

Similarly we choose a subset Ω_L having size $\overline{Z}$ and containing the firms with the highest values of $\underline{V}_i$. Then we identify V_L as $\min\{\underline{V}_i : i \in \Omega_L\}$ so that $\underline{V}_i \geq V_L$ for $i \in \Omega_L$ and $\underline{V}_i \leq V_L$ for $i \notin \Omega_L$. We will show that no bid below V_L can be accepted in equilibrium. For a well-behaved equilibrium, where each firm bids its value, V_L would be the lowest realized clearing price. We define $\Omega'_L = \Omega_L \backslash \{j_L\}$ where $j_L \in \Omega_L$ is a firm with $\underline{V}_{j_L} = V_L$. We define $V'_L = \min\{\underline{V}_i : i \in \Omega'_L\}$. Thus $V'_L \geq V_L$ and they may be the same. We will show that any bid above V'_L will be accepted with a positive probability.

Note that even if the sets Ω_U etc. are not uniquely defined, due to common upper or lower bounds of value ranges, the values of V_U, V'_U, V_L, V'_L are still fixed. These values are illustrated in Figure 1. This shows a set of bids (on the vertical axis) for four firms with different ranges of values (on the horizontal axis). We

[35] In case of a single-object auction with two bidders, the reflected auction would also be a single-object auction with two bidders. In this special case, both upper and lower partial high-low equilibria can occur in the same auction. Moreover, partial high-low equilibria could occur for any subrange of values and there could be multiple subranges where partial high-low equilibria occur in the same equilibrium.

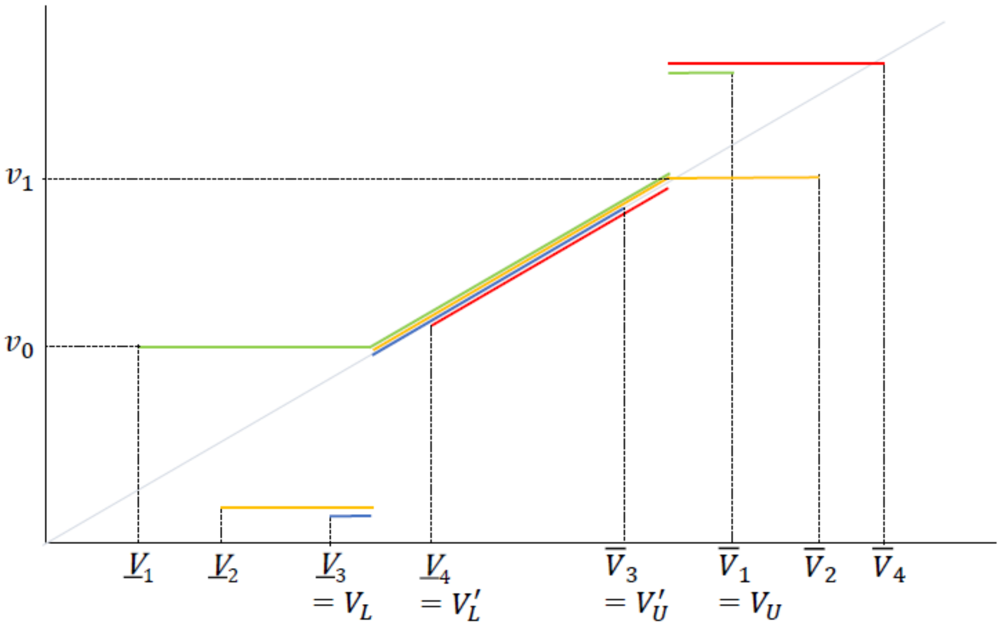

Figure 1: Example where bidders have asymmetric value ranges. A possible equilibrium is shown for the case $\underline{Z} = \overline{Z} = 2$.

will show later that this is an equilibrium set of bids if $v_0 \in [V_L, V_L']$, $v_1 \in [V_U', V_U]$, and any price cap or price floor is non-restrictive.

Blume & Heidhues (2004) consider the case with at least three bidders in a single-object auction. In our setting this corresponds to the case where at least two bids are rejected in every auction. We also make the reflected version of this assumption.

Assumption A: $n \geq \overline{Z} + 2$ and $\underline{Z} \geq 2$.

Without this assumption there would, similar to Blume & Heidhues (2004), Blume et al. (2009) and Corollary 2, exist partial high-low equilibria. We avoid them in most of the price range, but they can show up at the edges.

We allow firms to have different value ranges, and therefore need to make sure that all of these n firms are relevant, i.e. that they have the possibility to compete at relevant prices. Hence, we also make assumptions on the value ranges of the n firms. We let $V_L'' = \min\{\overline{V}_i\}$ and $V_U'' = \max\{\underline{V}_i\}$, and we assume:

Assumption B: $V_L'' > V_L'$ and $V_U'' < V_U'$.

By definition $V_L'' \leq V_U'$ and $V_U'' \geq V_L'$ so this assumption establishes the ordering $V_L \leq V_L' < V_U' \leq V_U$. Note that a firm with $\overline{V}_i \leq V_L$ would not take part in the auction. Also a firm with $\underline{V}_i \geq V_U$ will always have a value high enough to be accepted, and could be allocated a unit before the auction starts. Assumption B rules out such outcomes.

We write X_x for the set of firms which can have a signal x, i.e. $X_x = \{i \,:\, x \in (\underline{V}_i, \overline{V}_i)\}$, and we will assume

Assumption C: The set X_x contains at least three firms for any value $x \in (V_L, V_U)$, with potentially different firms for different x values.

7.4.1 Ex-post optimality

In the private value model, equilibrium bids would not be changed even if other player's bids are known (i.e. ex-post optimality). The only outcomes that would make a bid not ex-post optimal involve ties that occur with measure zero.[36]

Lemma 13 *With private values, each bid is almost surely ex-post optimal in an equilibrium.*

7.4.2 High-low equilibrium

Next, we will study the high-low equilibrium. If the auctioneer's supply is certain, then there is an asymmetric equilibrium where some firms always bid high and others always bid low.

Proposition 9 *If the auctioneer's supply is certain so that* $Z = \underline{Z} = \overline{Z}$*, and a set* Ω *of exactly* Z *high bidding firms have bids at least as high as* $\max\{\overline{V}_i : i \notin \Omega\}$ *and the remaining low bidding firms all have bids that are no larger than* $\min\{\underline{V}_i : i \in \Omega\}$*, then this is a BNE (the high-low equilibrium).*

Under our assumptions, it can be shown that there is no other equilibrium where the bid of a firm is rejected with probability 1 for its highest signal or where the bid of a firm is accepted with probability 1 for its lowest signal. Moreover, high-low equilibria only occur when the auctioneer's supply is certain.

Lemma 14 *If the equilibrium bid of a firm is rejected with probability* 1 *for the highest signals, then supply must be certain and the equilibrium must be of the high-low type. If the equilibrium bid of a firm is accepted with probability* 1 *for the lowest signals, then supply must be certain and the equilibrium must be of the high-low type.*

7.4.3 Partial high-low equilibria

We will show that the remaining equilibria are well-behaved or partly well-behaved. For these equilibria, all firms will bid their value for some mid-range of signals (v_0, v_1). But bidding might be ill behaved near the edges, in the intervals $[V_L, V_L']$ and $[V_U', V_U]$, where partial high-low equilibria exist.

[36] The result is true also for single-object auctions and for the reflected version of single-object auctions.

Theorem 1 *If an equilibrium is not of high-low type, then it must have the following properties: (a) There are points $v_0 \in [V_L, V'_L]$ and $v_1 \in [V'_U, V_U]$ such that for all signals in (v_0, v_1) all the firms bid at their values. (b) When $v_0 = V_L$ then all firms bid at v_0 or lower for signals strictly less than v_0. (c) When $v_0 > V_L$ there is a single firm, i_X, which bids at v_0 for signals less than v_0, while the other firms bid at a value $\underline{V}_{i_X}$ or lower for signals strictly less than v_0. (d) When $v_1 = V_U$ then all firms bid at any value at v_1 or higher for signals greater than v_1. (e) When $v_1 < V_U$, then one firm i_Y with $\overline{V}_{i_Y} > v_1$ bids at v_1 for almost all signals in $(v_1, \overline{V}_{i_Y})$ while the other firms bid at any value at $\overline{V}_{i_Y}$ or higher for signals strictly greater than v_1. Moreover, a set of bid functions that satisfies properties (a) - (e) constitutes an equilibrium.*

For equilibria with $v_0 > V_L$, a single bidder i_X bids high, at v_0, for signals less than v_0. Other bidders bid low, at or below $\underline{V}_{i_X}$, for signals less than v_0. Hence, for signals less than v_0, the equilibrium is similar to the high-low equilibrium at the bottom. If $v_1 < V_U$ then there is a high-low equilibrium at the top. In this case, a single bidder i_y bids low, at v_1, for signals higher than v_1. Other bidders bid high, at or above $\overline{V}_{i_Y}$, for signals higher than v_1. Figure 1 illustrates the partial high-low equilibria at the top and bottom. Partial high-low equilibria have outcomes where a bid is accepted and another bid is rejected even if the latter bidder has a higher valuation, which is inefficient. But inefficiencies only occur for values in the ranges $[V_L,\ v_0]$ and $[v_1, V_U]$, where bidders do not bid their value.

In case bidders have the same value range, as assumed in Blume & Heidhues (2004), we get $v_0 = V_L = V'_L = \underline{V}$ and $v_1 = V'_U = V_U = \overline{V}$, so that Theorem 1 simplifies as follows:

Corollary 3 *If all bidders have the same range of private values $\left(\underline{V}, \overline{V}\right)$, then there is an efficient equilibrium where every bidder bids at its value for each signal. This is the unique equilibrium if Z can take more than one value. But in the case when Z is fixed, then high-low equilibria also exist.*

Recall that in our setting, we have chosen the number of objects Z such that it is strictly larger than one and such that at least two bids are always rejected. These assumptions rule out the single-object auction studied by Blume & Heidhues (2004), and the reflected version of that setting. This is why there are no partial high-low equilibria in our setting when all bidders have the same value range.

7.4.4 Extension 1: Price floor and price cap

There is no price cap and no price floor (reservation price) in the model analysed above. In this extension we will argue that they can be used to give a unique equilibrium. We omit formal proofs, but arguments below could be formalized using minor variations of the proofs used in Sections 7.3 and 7.4.3.

We will first consider the case where all bidders have the same range of private values $\left(\underline{V}, \overline{V}\right)$. For this case, it can be shown that an effective price floor would give a unique equilibrium. Any price floor $\underline{p}$ in the range $\left(\underline{V}, \overline{V}\right)$ would knock out high-low equilibria, also when the volume Z is certain. The reason is that the market price is at least $\underline{p}$, so it is no longer profitable for high bidders to bid high, and get accepted with certainty, for signals in the range $\left(\underline{V}, \underline{p}\right)$. This also means that low bidders, which have zero profit in a high-low equilibrium, will be accepted with a positive probability and make a positive profit if they deviate and bid their value for signals in the range $\left(\underline{p}, \overline{V}\right)$.

It appears that this has not been discussed in the previous literature, but an effective price cap would also give uniqueness if all bidders have the same range of private values $\left(\underline{V}, \overline{V}\right)$. Any price cap $\overline{p}$ in the range $\left(\underline{V}, \overline{V}\right)$ would knock out high-low equilibria, also when the volume Z is certain. The reason is that if high bidders bid at $\overline{p}$ (or lower), then it would be profitable for low bidders to deviate, and bid at $\overline{p}$, when observing signals in the range $(\overline{p}, \overline{V})$. Such a bid would be accepted with a positive probability and give a positive payoff in case of acceptance. Knocking out the high-low equilibrium implies that there are circumstances where introducing a maximum price would actually increase the revenue of the auctioneer.

The analysis in Section 7.4.3 is based on Assumption A, which rules out single-object auctions. The latter case is special. As argued in Section 7.3, high-low equilibria at the bottom exist in a single-object auction, even if all bidders have the same range of private values. Blume & Heidhues (2004) show that an effective price floor would knock out such an equilibrium and give uniqueness in such an auction. We realize that a price cap does not have any effect on the lower partial high-low equilibrium. Hence, an efficient price cap is not sufficient to give uniqueness in a single-object auction, even if all bidders have the same range of private values. This is the other way around for the reflected version of the single-object auction. In the reflected auction, an effective price cap (and not a price floor) is sufficient to give uniqueness.

It gets more complicated when bidders have different value ranges. As shown in Theorem 1, high-low equilibria can occur at the bottom if firms have different lower bounds on their values. One needs a price floor to knock out this partial high-low equilibrium. Any price floor $\underline{p}$ in the range (V_L, V_L') would prevent i_X from bidding high, at v_0, for signals less than $\underline{p}$, and the partial high-low equilibrium falls apart. Such a price floor would also knock out any high-low equilibrium. A price floor in the range (V_L, V_L') would give uniqueness if bidders have a common upper bound on their values.

If firms have different upper bounds on their values, then it follows from Theorem 1 that partial high-low equilibria can occur at high prices. Such an equilibrium can be knocked out by a price cap. Any price cap in the range (V_U', V_U) would prevent firms from bidding high, at or above $\overline{V}_{i_Y}$, for signals above $\overline{p}$. Such a price cap would also knock out any high-low equilibrium. A price cap in the range (V_U', V_U) would give uniqueness if bidders have a common lower bound on their values. If bidders have both different upper and lower bounds on their values, then both a price floor and a price cap are needed to get uniqueness.

7.4.5 Extension 2: Price-sensitive supply

Another way to knock out ill-behaved equilibria and to ensure a unique equilibrium is to make the auctioneer's supply price sensitive. One way to introduce such sensitivity is to let the auctioneer make bids, which allows the auctioneer to buy back units if its own bids are accepted.

If all bidders (excluding the auctioneer) have the same range of private values $\left(\underline{V}, \overline{V}\right)$ and Assumption A is satisfied, then multiple equilibria can only occur when the auctioneer's supply is certain. Any additional equilibrium would be a high-low equilibrium (see Corollary 3). In this case, it is enough to knock out the high-low equilibrium to ensure uniqueness. Such an equilibrium can be ruled out if the auctioneer has a bid at $p_0 \in \left(\underline{V}, \overline{V}\right)$. In a potential high-low equilibrium, such a bid would be accepted with a positive probability or rejected with a positive probability. In the first case, a low bidder (with bids at $\underline{V}$) will find it profitable to deviate and bid at value when observing values above p_0. In the second case, a high bidder (with bids at $\overline{V}$) will find it profitable to deviate and bid at value when observing values below p_0.

In order to keep sales and payoff high, an auctioneer would normally prefer to make a bid just above $\underline{V}$. In our model with single-unit demand this would essentially have the same effect as a price floor just above $\underline{V}$.[37] Moreover, similar to the price floor, a single bid just above $\underline{V}$ would get rid of the high-low equilibrium at the bottom that can occur in a single-object auction, and give uniqueness also in such an auction.

Similarly, a producer can avoid the risk of getting ill-behaved equilibria by making a bid at a high price. The argument above would work for any $p_0 \in \left(\underline{V}, \overline{V}\right)$. Reducing its supply when the price is high could be costly for the auctioneer, but it could be attractive if it could find an extra unit to sell when the price is high. In particular, selling an extra unit above p_0, would rule out the high-low equilibrium at the top, and ensure uniqueness, in the reflected version of the single-object auction. But if an auctioneer is not able to find an extra unit when the price is high, then a price cap would be a better alternative.

Similarly, the auctioneer can use price-sensitive supply to ensure a unique equilibrium in an auction where bidders have different value ranges. If Assumptions A-C are satisfied, it can be shown that if the auctioneer makes a high bid in the range (V_U', V_U) and makes a low bid in the range (V_L, V_L'), then this would give a unique equilibrium, which is well-behaved.[38] But if an auctioneer is not able to find an extra unit when the price is high, then a price cap in the range (V_U', V_U) would be a better alternative.

[37] There can be minor differences in revenue from unlikely events where several bidders observe values just above $\underline{V}$.

[38] The definition of the number of units, and accordingly also the definitions of V_U', V_U, V_L and V_L', becomes somewhat unclear when the auctioneer's supply is sensitive to the price. In this dicussion, we let n be the number of units that are traded when prices are in the mid range (V_L', V_U'). Hence, $n + 1$ units are traded when the price is high and $n - 1$ units when the price is low.

8 Conclusion

In this paper, we consider uniform-price, sales auctions where the highest rejected bid sets the clearing price. We make a detailed study of a uniform-price auction where bidders have single-unit demand and asymmetric information. We consider two models. In the first, we assume that signals are affiliated as in Milgrom & Weber (1982) and Weber (1983). Our contribution is that we allow bid functions to be non-monotonic and discontinuous, and the auctioneer's supply to be uncertain. Moreover, the auctioneer's supply is allowed to be correlated with signals of the bidders. A well-behaved symmetric and monotonic equilibrium exists as long as an affiliation condition is satisfied. We show that Milgrom & Weber's (1982) and Weber's (1983) results for the ranking of symmetric, monotonic equilibria in uniform-price and pay-as-bid auctions continue to hold under this affiliation condition, and so does the publicity effect. Under somewhat more restrictive conditions, where the auctioneer's supply is independent of bidders's signals, we are able to solve for all symmetric equilibria. We show that there is exactly one symmetric equilibrium, which is monotonically increasing.

In practice, uniform-price auctions sometimes have ill-behaved equilibria. We study this problem for a model with private values. We are able to solve for all pure-strategy equilibria, including asymmetric equilibria. We show that equilibria with prices at the collusive level exist, also when the number of bidders is large. We refer to them as high-low equilibria, because some bidders always bid high and others always bid low. The high bids are always accepted and the low bids are always rejected. The highest low bid sets the price. We show that uncertainty in the auctioneer's supply removes this problem. If the auctioneer's supply is uncertain (a tiny uncertainty is sufficient) and bidders are symmetric ex ante, then there is a unique equilibrium, which is well-behaved. If supply is certain, then an effective price floor or price cap will give a unique equilibrium. Hence, there are circumstances where introducing a maximum price can actually increase the revenue of an auctioneer that is selling goods.

Single-object auctions are special. In a second-price auction, bidders can play high-low equilibria for the lower part of the value range. In this case, an effective price floor, but not a price cap, gives uniqueness, if there are at least three bidders and all of them have the same value range. Another special case is when there is exactly one more bidder than the number of objects. In this case it is the other way around. Bidders can play high-low equilibria for the upper part of the value range. An effective price cap gives a unique equilibrium, but not a price floor.

For the case with at least two objects and at least two more bidders than objects, we also consider bidders with different value ranges. In this case, bidders can play high-low equilibria for part of the value range, when some bidders have hit their upper or lower bound. An effective price cap is necessary to get a unique equilibrium if bidders have different upper bounds on values. If bidders have different lower bounds on values, then an effective price floor is necessary for uniqueness. If bidders have both different lower and upper bounds, then both an effective price floor and an effective price cap are needed to get a unique equilibrium. Unique equilibria are well-behaved and efficient.

A small elasticity with respect to the price in the auctioneer's supply can also be efficient in reducing the set of equilibria. For example, an auctioneer can make a bid at low prices to buy back a unit of the good. In our model with single-unit demand this has the same effect as a price floor at the same price. Price-sensitive supply at high prices can be of interest if the auctioneer is able to find an extra unit to sell at high prices. Otherwise price-sensitive supply at high prices can be costly for the auctioneer. If so, it is better to have a price cap at high prices instead.

When analysing uniform-price auctions, we noticed that the auction has an invariance property. If there is an equilibrium in an auction with n bidders and Z goods, then there is a corresponding equilibrium in a transformed auction with n bidders and $n - Z$ goods, if the sign of all values and bids are reversed. We refer to the transformed auction as the reflected auction. As an example, the reflected version of a single-object, sales auction with n bidders is a sales auction with n bidders and $n - 1$ goods. This relationship explains why a price floor is needed to get uniqueness in the single-object auction and a price cap in the reflected version of this auction.

Our study considers the case where bidders have single-unit demand, which gives them limited market power. If bidders have significant market power, then a large elasticity with respect to the price or large uncertainty in the supply of the auctioneer is likely to be needed to avoid prices at the collusive level in uniform-price auctions. Similarly, if bidders have significant market power, then price caps and price floors would have to be more restrictive than in our model to ensure a unique equilibrium. Moreover, in case of significant market power, an auctioneer might also find it useful to have elastic supply in the whole price range, and not only at low or high prices.

9 References

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Appendix

Proofs and technical lemmas of Section 3

The following lemma shows a diagonal property of the density function which will be useful in some of our proofs.

Lemma 15 *If $f_k(\boldsymbol{S}, q_0, x_2, x_3, ..., x_n) > 0$ for some values $x_2, x_3, ..., x_n$ and $\boldsymbol{S}$, then $f_k(\boldsymbol{S}, q_0, q_0, ..., q_0) > 0$. Moreover for every $\mathbf{S} \in Q_S$ and $x \in Q_X$ there is a k such that $f_k(\mathbf{S}, x, x, ..., x) > 0$.*

Proof. Given $f_k(\boldsymbol{S}, q_0, x_2, x_3, ..., x_n) > 0$ then by symmetry of f_k in Assumption 4, we deduce that $f_k(\boldsymbol{S}, x_2, q_0, x_3, ..., x_n) > 0$ and hence from the affiliation property (1), $f_k(\boldsymbol{S}, q_0, q_0, x_3, ..., x_n) > 0$. Then by symmetry $f_k(\boldsymbol{S}, q_0, x_3, q_0, x_4, ..., x_n) > 0$ and again the affiliation property shows that $f_k(\boldsymbol{S}, q_0, q_0, q_0, x_4, ..., x_n) > 0$. Continuing in this way we can establish that $f_k(\boldsymbol{S}, q_0, q_0, ..., q_0) > 0$. The final statement follows from this and our assumption that for every $\mathbf{S} \in Q_S$ and $x \in Q_X$, the density $f_k(\mathbf{S}, x, \mathbf{X}_{-1}) > 0$ for some $\mathbf{X}_{-1}$ and k. ∎

Lemma 16 is a technical result, which will be useful when proving Proposition 1.

Lemma 16 *If $a_1 \geq a_2 \geq ...a_k$ and $b_1 \geq b_2 \geq ...b_k$ then for any set of probabilities q_j, $j = 1, ..., k$ with $q_j \geq 0$ and $\sum_{j=1}^{k} q_j = 1$,*

$$\sum_{j=1}^{k} q_j a_j b_j \geq \left(\sum_{j=1}^{k} q_j a_j\right)\left(\sum_{j=1}^{k} q_j b_j\right).$$

Proof. Let $\overline{a} = \sum_{j=1}^{k} q_j a_j$ and choose h such that $a_h \geq \overline{a} \geq a_{h+1}$. Then

$$\sum_{j=1}^{k} q_j \left(a_j - \overline{a}\right)\left(b_j - b_h\right) \geq 0$$

since the products involved are all between two elements of the same sign. Hence

$$\sum_{j=1}^{k} q_j a_j b_j - b_h \sum_{j=1}^{k} q_j a_j - \overline{a}\left(\sum_{j=1}^{k} q_j b_j\right) + b_h \overline{a}\left(\sum_{j=1}^{k} q_j\right) \geq 0.$$

Two of the terms are equal to $b_h\overline{a}$ and $-b_h\overline{a}$, respectively. They cancel out, which gives the inequality we require. ∎

Proof. (Proposition 1) We write T for the $m+n$-tuple $S_1, ..., S_m, X_1, Y_1, ...Y_{n-1}$, then from Milgrom & Weber Theorem 2 the variables in T are affiliated. We want to show that T, Y_Z is affiliated. To do this we will consider arbitrary increasing sets A and B in R^{m+n+1}, as well as an arbitrary sublattice L in R^{m+n+1}.

We define the maps $\eta_i : R^{m+n} \to R^{m+n+1}$ for $i = 1, ..., n-1$ by $\eta_i(u) = (u, u_{m+i+1})$, so that when $u = (S_1, ..., S_m, X_1, Y_1, ...Y_{n-1})$ then

$$\eta_i(u) = (S_1, ..., S_m, X_1, Y_1, ...Y_{n-1}, Y_i).$$

We define for any set U in R^{m+n+1}

$$U_i' = \{u \in \mathbb{R}^{m+n} \ : \ \eta_i(u) \in U\}.$$

for $i = 1, ..., n-1$. Note that with this definition we have that A_i' and B_i' are increasing sets in R^{m+n} and also $L_i' = \{u \in R^{m+n} \ : \ \eta_i(u) \in L\}$ is a sublattice in R^{m+n}.

Now $(T, Y_i) \in A$ if and only if $T \in A_i'$. Thus we can use the affiliation property for T to show that

$$\begin{aligned}\Pr((T,Y_i) \in A \cap B \mid L) &= \Pr(T \in A_i' \cap B_i' \mid L_i') \\ &\geq \Pr(T \in A_i' \mid L_i') \Pr(T \in B_i' \mid L_i') \\ &= \Pr((T,Y_i) \in A \mid L) \Pr((T,Y_i) \in B \mid L). \quad (10)\end{aligned}$$

In this inequality we have both left and right hand sides equal to zero in the case that $L_i' = \emptyset$. Now from the independence of Z,

$$\Pr((T,Y_Z) \in A \cap B \mid L) = \sum_{i=1}^{n-1} \Pr(Z = i) \Pr((T,Y_i) \in A \cap B \mid L).$$

Thus using (10), we deduce

$$\Pr((T,Y_Z) \in A \cap B \mid L) \geq \sum_{i=1}^{n-1} \Pr(Z = i) \Pr((T,Y_i) \in A \mid L) \Pr((T,Y_i) \in B \mid L). \quad (11)$$

In order to use Lemma 16 on this sum, we require $\Pr((T,Y_i) \in A \mid L)$ decreasing in i. But, since $Y_i \geq Y_{i+1}$, and A is increasing, if $(T, Y_{i+1}) \in A$ then $(T, Y_i) \in A$. Hence

$$\Pr((T,Y_i) \in A \mid L) \geq \Pr((T,Y_{i+1}) \in A \mid L).$$

Similarly $\Pr((T,Y_i) \in B \mid L) \geq \Pr((T,Y_{i+1}) \in B \mid L)$. Then we apply Lemma 16 and obtain from (11):

$$\begin{aligned}\Pr((T,Y_Z) \in A \cap B \mid L) &\geq \sum_{i=1}^{n-1} \underbrace{\Pr(Z = i)}_{q_i} \underbrace{\Pr((T,Y_i) \in A \mid L)}_{a_i} \underbrace{\Pr((T,Y_i) \in B \mid L)}_{b_i} \\ \geq\ & \left(\sum_{i=1}^{n-1} \Pr(Z = i) \Pr((T,Y_i) \in A \mid L)\right) \left(\sum_{i=1}^{n-1} \Pr(Z = i) \Pr((T,Y_i) \in B \mid L)\right) \\ =\ & \Pr((T,Y_Z) \in A \mid L) \Pr((T,Y_Z) \in B \mid L).\end{aligned}$$

Since A and B are arbitrary increasing sets in $\mathbb{R}^{m+n+2}$, this demonstrates the inequality we need to show that the variables $S_1, ..., S_m, X_1, Y_1, ...Y_{n-1}, Y_Z$ are affiliated. ■

Proofs and technical lemmas of Section 4

Lemma 17 *Suppose we have sets C_i, $i = 1, 2, ..., n$ where C_i is a collection of h_i disjoint intervals in Q_X each of which is either $(a_j^{(i)}, b_j^{(i)})$, $[a_j^{(i)}, b_j^{(i)})$, $(a_j^{(i)}, b_j^{(i)}]$ or $[a_j^{(i)}, b_j^{(i)}]$ with $a_j^{(i)} \leq b_j^{(i)} < a_{j+1}^{(i)}$ for $j = 1, 2, ..., h_i$ and where $a_1^{(i)} \in \mathbb{R} \cup \{-\infty\}$ and $b_{h_i}^{(i)} \in \mathbb{R} \cup \{+\infty\}$, and we let $\boldsymbol{C} = C_1 \times C_2 ... \times C_n$. Suppose that for signal $\boldsymbol{S}$ there is $\boldsymbol{x}^0 \in \boldsymbol{C}$, with $f_k(\boldsymbol{S}, \boldsymbol{x}^0) > 0$ for some k. Then, $\mathbb{E}[\, V_1 |\, \boldsymbol{S}, \boldsymbol{X} \in \boldsymbol{C}]$ is continuous in each of the arguments $a_j^{(i)}, b_j^{(i)}$, $j = 1, 2, ..., h_i$, $i = 1, 2, ..., n$, if $a_j^{(i)} < b_j^{(i)}$, and continuous in $\overline{a}_j^{(i)}$ if $a_j^{(i)} = b_j^{(i)} = \overline{a}_j^{(i)}$. Moreover the limit of $\mathbb{E}[V_1 |\, \boldsymbol{S}, \boldsymbol{X} \in \boldsymbol{C}]$ as $a_j^{(i)} \to b_j^{(i)}$ is equal to its value when $a_j^{(i)} = b_j^{(i)}$.*

Proof. First consider the case where $\Pr(\mathbf{S}, \mathbf{X} \in \mathbf{C}) > 0$. Then

$$\mathbb{E}[\, V_1 |\, \boldsymbol{S}, \boldsymbol{X} \in \boldsymbol{C}] = \frac{\sum_{k \in K_Z} \Pr(Z = k) \int_{\boldsymbol{X} \in \boldsymbol{C}} u\left(\boldsymbol{S}, X_1, \boldsymbol{X}_{-1}\right) df_k\left(\boldsymbol{S}, \boldsymbol{X}\right)}{\sum_{k \in K_Z} \Pr(Z = k) \int_{\boldsymbol{X} \in \boldsymbol{C}} df_k\left(\boldsymbol{S}, \boldsymbol{X}\right)}.$$

We have continuity of $\mathbb{E}[\, V_1 |\, \boldsymbol{S}, \boldsymbol{X} \in \boldsymbol{C}]$ when considered as a function of $a_j^{(i)}$ and $b_j^{(i)}$ since both $\int_{\boldsymbol{X} \in \boldsymbol{C}} u\left(\boldsymbol{S}, X_1, \boldsymbol{X}_{-1}\right) df_k\left(\boldsymbol{S}, \boldsymbol{X}\right)$ and $\int_{\boldsymbol{X} \in \boldsymbol{C}} df_k\left(\boldsymbol{S}, \boldsymbol{X}\right)$ are continuous functions of these parameters. In the case that $\Pr(S, X \in C) = 0$, and so $\int_{\boldsymbol{X} \in \boldsymbol{C}} df_k\left(\boldsymbol{S}, \boldsymbol{X}\right) = 0$ for each k, then we must have $a_j^{(i)} = b_j^{(i)} = \overline{a}_j^{(i)}$ for all $j = 1, 2, ..., h_i$ for i in some set I_0. In this case we replace integrals with sums for X_i when $i \in I_0$ in the definition of $\mathbb{E}[\, V_1 |\, \boldsymbol{S}, \boldsymbol{X} \in \boldsymbol{C}]$. For example when $n = 2$ and $a_j^{(1)} = b_j^{(1)}$ for all $j = 1, 2, ..., h_1$ we have

$$\mathbb{E}[\, V_1 |\, \boldsymbol{S}, \boldsymbol{X} \in \boldsymbol{C}] = \frac{\sum_{k \in K_Z} \Pr(Z = k) \sum_{j=1}^{h_1} \int_{(\overline{a}_j^{(1)}, X_2) \in \boldsymbol{C}} u\left(\boldsymbol{S}, a_j^{(1)}, X_2\right) df_k\left(\boldsymbol{S}, a_j^{(1)}, X_2\right)}{\sum_{k \in K_Z} \Pr(Z = k) \sum_{j=1}^{h_1} \int_{(\overline{a}_j^{(1)}, X_2) \in \boldsymbol{C}} df_k\left(\boldsymbol{S}, a_j^{(1)}, X_2\right)}.$$

Note that the integrals here are with respect to X_2 with $\overline{a}_j^{(1)}$ fixed. Thus from our assumption that there is $x^0 \in C$, with $f_k(S, x^0) > 0$ for some k we know that at least one of these integrals has non-zero value. When $I_0 = \{1, 2, \ldots, N\}$ then there are no integrals in the expression and the denominator becomes the sum over the non-zero $f_k(S, x)$ values for $x \in C$. The conclusion on continuity still holds in this case. Also the final statement of the Lemma follows directly from the continuity of f_k and u. ■

Note that the set $\mathbf{C}$ that was defined in Lemma 17 constitutes a sublattice.

Proof. (Lemma 1) We use a similar argument to Theorem 5 in Milgrom & Weber (1982). We will follow the notation of Milgrom & Weber, so in this proof we do not have the usual meanings for S and Z. Given the sublattice L consider an arbitrary $a_1 < a_2$, from which we define a sublattice $S = \{(x_1, \widetilde{\boldsymbol{X}}) \;:\; a_1 \leq x_1 \leq a_2,\ \widetilde{\boldsymbol{X}} \in L\}$. Theorem 23 in Milgrom & Weber states that if $Z_1, ..., Z_k$ are affiliated, then for every non-decreasing function g, increasing set A, and sublattice S, $\mathbb{E}[g(Z)|\; Z \in A \cap S] \geq \mathbb{E}[g(Z)|\; Z \in S] \geq \mathbb{E}[g(Z)|\; Z \in A^c \cap S]$, where A^c is the complement of A. Since V_1 is a non-decreasing function of the affiliated variables $X_1, X_2 ... X_n$, we can use $A = \{X_1 \geq a_1 + \delta\}$ for $\delta > 0$, and then applying this result (Theorem 23 of Milgrom & Weber) shows that $\mathbb{E}[V_1|\ a_1 \leq X_1 \leq a_2, \widetilde{\mathbf{X}} \in L]$ increases when a_1 is replaced by $a_1+\delta$, and hence is an increasing (non-decreasing) function of the left-hand end of the interval. Similarly setting $A = \{X_1 \geq a_2 - \delta\}$ and considering A^c shows that $\mathbb{E}[V_1|\ a_1 \leq X_1 \leq a_2, \widetilde{\mathbf{X}} \in L]$ is decreased when a_2 is replaced by $a_2 - \delta$, and hence is an increasing function of the right- hand end of the interval. Thus $\mathbb{E}[V_1|\ a_1 \leq X_1 \leq a_2, \widetilde{\mathbf{X}} \in L]$ is increasing in both a_1 and a_2. We can use the final statement of Lemma 17 to show that there is continuity at the limit as $a_1 \to a_2$. This establishes that $\mathbb{E}[V_1|\, X_1 = x, \widetilde{\mathbf{X}} \in L]$ is increasing (non-decreasing) with respect to x.

To establish strictly increasing, take any two possible signals $x_L < x_U$ and both in Q_X. Define V_1^L from V_1 by setting $V_1^L(\boldsymbol{S}, \boldsymbol{X}) = V_1(\boldsymbol{S}, x_L, X_2, ..., X_n)$ for $X_1 \in [x_L, x_U]$ and $V_1^L = V_1$ otherwise. Thus using the fact that u is strictly increasing, we have $V_1^L < V_1$ for $X_1 = x_U$. Then

$$\begin{aligned} \mathbb{E}\left[V_1|\, X_1 = x_L, \widetilde{\mathbf{X}} \in L\right] &= \mathbb{E}\left[V_1^L|\, X_1 = x_L, \widetilde{\mathbf{X}} \in L\right] \\ &\leq \mathbb{E}\left[V_1^L|\, X_1 = x_U, \widetilde{\mathbf{X}} \in L\right] \\ &< \mathbb{E}\left[V_1|\, X_1 = x_U, \widetilde{\mathbf{X}} \in L\right] \end{aligned}$$

which proves the result. ■

Proof. (Lemma 2) We consider the value of $\mathbb{E}[V_1|\, X_1 = x, W_Z = b_0, Z = k]$, for $k \in K_Z$. By symmetry we can suppose that bidder 2 has the kth highest bid amongst the bidders other than bidder 1, and condition on that it bids b_0. Moreover, it can be assumed that bidders $3, 4, ..., k+1$ all bid at or above b_0, and the other bids from bidders $k+2, \ldots, n$ are at b_0 or below. Hence

$$\mathbb{E}[V_1|\, X_1 = x, W_Z = b_0, Z = k] = \mathbb{E}\left[V_1|\, X_1 = x, (X_2, X_3, ..., X_n) \in H_k(b_0), Z = k\right],$$

where

$$H_k(b_0) = \{(x_2, ..., x_n)\ :\ b^*(x_2) = b_0,\ b^*(x_i) \geq b_0, i = 3, ..., k+1,\ b^*(x_j) \leq b_0, j = k+2, ..., n\}.$$

Notice that $H_k(b_0)$ is a sublattice and is not empty because if $b^*(x_0) = b_0$ then the point $x_2 = x_3 = ... = x_n = x_0$ is in $H_k(b_0)$. Thus from Lemma 1, $\mathbb{E}[V_1|\, X_1 = x, W_Z = b_0, Z = k]$ is strictly increasing in x. If Z is independent, then

$$\mathbb{E}[\, V_1|\, X_1 = x, W_Z = b_0] = \sum_{k \in K_Z} \Pr(Z = k) \mathbb{E}[V_1|\, X_1 = x, W_Z = b_0, Z = k],$$

so $\mathbb{E}[V_1|\, X_1 = x, W_Z = b_0]$ is strictly increasing in x in this case. ■

Proof. (Lemma 3) Suppose that there is an interval $(a^{(u)}, a^{(u+1)})$ on which b^* is constant, say $b^*(x) = b_0$ on this interval. There may be other such intervals and we write $L_0 \subset R$ for the set $\{x \mid b^*(x) = b_0\}$. Observe from our assumption that for any $\boldsymbol{S}$, $f_k(\boldsymbol{S}, x, \mathbf{X}_{-1}) > 0$ for some $\mathbf{X}_{-1}$ and $k \in K_Z$ for each x in $(a^{(u)}, a^{(u+1)})$. Thus from Lemma 15 we can fix a value $q_0 \in (a^{(u)}, a^{(u+1)})$ with $f_k(\boldsymbol{S}, q_0, q_0, ..., q_0) > 0$. We set $K_0 = \{k \in K_Z : f_k(\boldsymbol{S}, q_0, q_0, ..., q_0) > 0\}$. We can deduce from the continuity of the density function f_k that there is a range

$[q_0-\delta, q_0+\delta] \subset (a^{(u)}, a^{(u+1)})$ so that $f_k(\boldsymbol{S}, x_1, x_2, ..., x_n) > 0$ for $x_i \in [q_0-\delta, q_0+\delta]$, $i = 1, 2, ..., n$, for each $k \in K_0$, and $f_k(\boldsymbol{S}, x_1, x_2, ..., x_n) = 0$ for $x_i \in [q_0-\delta, q_0+\delta]$, $i = 1, 2, ..., n$ for $k \notin K_0$. Since $b^*(x) = b_0$ for $x \in [q_0-\delta, q_0+\delta]$ there is a non-zero probability, which we write as $p_0^{(k)}$, of all the bidders making the same bid b_0.

We define $p_0 = \Pr(W_Z = b_0 | X_1 \in (q_0, q_0+\delta])$ and $p_0' = \Pr(W_Z = b_0 | X_1 \in [q_0-\delta, q_0)$ which are both positive because they also include the case when all bidders make a bid of b_0.

From Lemma 2 we can deduce that $\mathbb{E}[V_1 | q_0 - \delta \le X_1 < q_0, W_Z = b_0] < \mathbb{E}[V_1 | q_0 < X_1 \le q_0+\delta, W_Z = b_0]$.[39] Thus we have two possible cases (A) $\mathbb{E}[V_1 | q_0 < X_1 \le q_0 + \delta, W_Z = b_0] > b_0$ or (B) $\mathbb{E}[V_1 | q_0 - \delta \le X_1 < q_0, W_Z = b_0] < b_0$.

In case (A) player 1 receives positive expected payoff for signals in the range $(q_0, q_0 + \delta]$ when $W_Z = b_0$. We consider changing the bid function for bidder 1 to $b_0 + \varepsilon$ for signals in this range. If a signal in this range occurs and $W_Z < b_0$ then both new and old bids are accepted and there is no change in payoff to player 1. If $W_Z > b_0$ then the old bid is not accepted and the new bid will be accepted only when $b_0 < W_Z \le b_0 + \varepsilon$ which occurs with a probability we define as $p_A(\varepsilon)$. The value of $\mathbb{E}[V_1 | q_0 < X_1 \le q_0 + \delta, b_0 < W_Z \le b_0 + \varepsilon]$ might be lower than $\mathbb{E}[V_1 | q_0 < X_1 \le q_0 + \delta, W_Z = b_0]$ but we can bound the difference by some constant Δ. Finally $W_Z = b_0$ with probability $p_0 > 0$ defined above. If this happens the old bid is accepted some of the time, while the new bid is always accepted. The probability of a bid of b_0 being accepted in this case depends on the number of competitors bidding at b_0, as well as the value of Z, and we write p_X for this probability. Note that $p_X < 1$. In the case that the bid b_0 is accepted there is no change in payoff from the change in bid. Hence the increase in profit is at least

$$p_0(1 - p_X)(\mathbb{E}[V_1 | q_0 < X_1 \le q_0 + \delta, W_Z = b_0] - b_0) - p_A(\varepsilon)\Delta.$$

But as $p_A(\varepsilon) \to 0$ when $\varepsilon \to 0$ this expression is strictly positive for $\varepsilon > 0$ chosen small enough and this contradicts the optimality of b^*.

In case (B) player 1 receives negative expected payoff for signals in the range $[q_0 - \delta, q_0)$ when $W_Z = b_0$. We consider changing the bid function for bidder 1 to $b_0 - \varepsilon$ for signals in this range. If a signal in this range occurs and $W_Z > b_0$ then neither new or old bids are accepted and there is no change in payoff to player 1. If $W_Z < b_0$ then the old bid is accepted and the new bid will also be accepted unless $b_0 > W_Z \ge b_0 - \varepsilon$. We write $p_B(\varepsilon)$ for the probability of this event, and as in case (A) we let Δ be the difference between $\mathbb{E}[V_1 | q_0 - \delta \le X_1 \le q_0, W_Z = b_0]$ and $\mathbb{E}[V_1 | q_0 - \delta \le X_1 \le q_0, b_0 > W_Z \ge b_0 - \varepsilon]$. Finally $W_Z = b_0$ with probability $p_0' > 0$ defined above. If this happens then the old bid is accepted some of the time, while the new bid is never accepted. We write p_X' for the probability of a bid of b_0 being accepted in this case. This might be different from p_X because of the different range for the signal X_1, but we have $p_X' > 0$. Hence the increase in

[39] [This follows because $\mathbb{E}[V_1 | q_0 - \delta \le X_1 < q_0, W_Z = b_0]$ is bounded above by $\sup_{q_0-\delta \le X_1 < q_0} \mathbb{E}[V_1 | X_1, W_Z = b_0]$ and $\mathbb{E}[V_1 | q_0 < X_1 \le q_0 + \delta, W_Z = b_0]$ is bounded below by $\inf_{q_0 < X_1 \le q_0+\delta} \mathbb{E}[V_1 | X_1, W_Z = b_0]$.

profit is at least

$$-p_0'p_X'(\mathbb{E}[V_1|\, q_0-\delta \leq X_1 < q_0, W_Z = b_0] - b_0) - p_B(\varepsilon)\Delta.$$

Thus we again have an increase in expected payoff for small enough ε by observing that $p_B(\varepsilon) \to 0$ when $\varepsilon \to 0$. This contradicts the optimality of b^*, hence establishing our result. ■

Proof. **(Lemma 4)** For continuity with respect to b we begin by fixing $k \in K_Z$. We condition on the set of bids above, below and at b. It will also be helpful to define $A(b) = \{x : b^*(x) > b\}$, $B(b) = \{x : b^*(x) < b\}$ so that these are collections of intervals defined by the bid function b^*.

We will consider the set of signals that would imply exactly $k-1$ bids (from the other bidders) strictly above b, and one bid at b. By symmetry we can suppose that bidder 2 bids at $\mathbf{b}$, and bidders $3, 4, ..., k+1$ are the $k-1$ bids above b with the other bids being below b. Thus

$$\mathbb{E}[V_1|\, X_1 = x, W_Z = b, Z = k] = \mathbb{E}\left[V_1|\, X_1 = x, (X_2, X_3, ..., X_n) \in H_k(b), Z = k\right] \tag{12}$$

where $H_k(b)$ is a sublattice defined by

$$H_k(b) = \{(x_2, x_3, ...x_n)\ :\ b^*(x_2) = b,\ x_3, x_4, ..., x_{k+1} \in A(b),\ x_{k+2}, x_{k+3}, ...x_n \in B(b)\} \tag{13}$$

is a set in $\mathbb{R}^{n-1}$. Notice that conditioning on $W_Z = b$ also allows for other components from $x_2, x_3, ..., x_n$ to take the value b, which are not included in our definition of $H_k(b)$. However since b^* is a regular equilibrium, Lemma 3 implies that each segment of b^* is either strictly increasing or decreasing. So the probability that $(X_2, X_3, ..., X_n) \in H_k(b)$ is positive but the probability that any of these variables are equal to b is zero. Thus the conditional expectation of (12) is correct.

Moreover because b is not in $G(b^*)$, and because the segments of b^* are either continuous increasing or continuous decreasing, a small change in b implies a small change in the end points of the intervals that make up $A(b)$ and $B(b)$, and hence from (13) small changes in the end points of intervals in $H_k(b)$. We deduce from Lemma 17 that $\mathbb{E}[V_1|\, X_1 = x, W_Z = b, Z = k]$ is continuous in b.

Finally we write v_W as follows:

$$v_W\,(x, b) = \sum_{k \in K_Z} \Pr(Z = k|\, X_1 = x, W_Z = b)\mathbb{E}[V_1|\, X_1 = x, W_Z = b, Z = k]. \tag{14}$$

So

$$\begin{aligned} & v_W\,(x, b) \\ = & \frac{\sum_{k \in K_Z} \Pr(Z = k) \Pr(X_1 = x, W_Z = b|\, Z = k)\mathbb{E}[V_1|\, X_1 = x, W_Z = b, Z = k]}{\sum_{k \in K_Z} \Pr(Z = k) \Pr(X_1 = x, W_Z = b|\, Z = k)} \\ = & \frac{\sum_{k \in K_Z} \Pr(Z = k) \Pr(X_1 = x, \mathbf{X}_{-1} \in H_k(b)|\, Z = k)\mathbb{E}[V_1|\, X_1 = x, W_Z = b, Z = k]}{\sum_{k \in K_Z} \Pr(Z = k) \Pr(X_1 = x, \mathbf{X}_{-1} \in H_k(b)|\, Z = k)}. \end{aligned}$$

Using the same argument on the way that changes in b cause changes in the set $H_k(b)$, we deduce that $v_W(x, b)$ is also continuous in x. ■

Proof. (Lemma 5) For a given signal $X_1 = x$, write $b_x^* = b^*(x)$. We will consider varying the bid b and we define

$$\Pi_1(b, x) = \mathbb{E}\left[(V_1 - W_Z)\, 1_{W_Z<b} \middle| X_1 = x\right]$$

to be the expected profit to bidder 1 from a bid b given a signal x.

Since b_x^* maximizes $\Pi_1(b, x)$ we have $\Pi_1(b_x^*, x) - \Pi_1(b_x^* - \delta, x) \geq 0$ and $\Pi_1(b_x^*, x) - \Pi_1(b_x^* + \delta, x) \geq 0$. Thus

$$\begin{aligned} & \mathbb{E}\left[(V_1 - W_Z)\, 1_{W_Z<b_x^*} - (V_1 - W_Z)\, 1_{W_Z<b_x^*-\delta} \middle| X_1 = x\right] \\ = \; & \mathbb{E}\left[(V_1 - W_Z)\, 1_{b_x^*-\delta\leq W_Z<b_x^*} \middle| X_1 = x\right] \geq 0. \end{aligned}$$

Thus

$$\mathbb{E}\left[(V_1 - b_x^* + \delta)\, 1_{b_x^*-\delta\leq W_Z<b_x^*} \middle| X_1 = x\right] \geq \mathbb{E}\left[(V_1 - W_Z)\, 1_{b_x^*-\delta\leq W_Z<b_x^*} \middle| X_1 = x\right] \geq 0.$$

Using Lemma 15 we know that there is a non-zero probability of all signals being in a range where bids are in the interval $(b_x^* - \delta, b_x^*)$. Hence $\Pr(b_x^* - \delta < W_Z < b_x^*) > 0$. Since

$$\begin{aligned} & \mathbb{E}\left[(V_1 - b_x^* + \delta)\, 1_{b_x^*-\delta\leq W_Z<b_x^*} \middle| X_1 = x\right] \\ = \; & \Pr(b_x^* - \delta < W_Z < b_x^*)\mathbb{E}\left[(V_1 - b_x^* + \delta) \middle| X_1 = x, b_x^* - \delta \leq W_Z < b_x^*\right], \end{aligned}$$

we can deduce that

$$\mathbb{E}\left[(V_1 - b_x^* + \delta) \middle| X_1 = x, b_x^* - \delta \leq W_Z < b_x^*\right] \geq 0. \tag{15}$$

Similarly

$$\begin{aligned} & \mathbb{E}\left[(V_1 - W_Z)\, 1_{W_Z<b_x^*} - (V_1 - W_Z)\, 1_{W_Z<b_x^*+\delta} \middle| X_1 = x\right] \\ = \; & -\mathbb{E}\left[(V_1 - W_Z)\, 1_{b_x^*\leq W_Z<b_x^*+\delta} \middle| X_1 = x\right] \geq 0. \end{aligned}$$

So

$$\mathbb{E}\left[(V_1 - b_x^* - \delta)\, 1_{b_x^*\leq W_Z<b_x^*+\delta} \middle| X_1 = x\right] \leq \mathbb{E}\left[(V_1 - W_Z)\, 1_{b_x^*\leq W_Z<b_x^*+\delta} \middle| X_1 = x\right] \leq 0,$$

and

$$\mathbb{E}\left[(V_1 - b_x^* - \delta) \middle| X_1 = x, b_x^* \leq W_Z < b_x^* + \delta\right] \leq 0. \tag{16}$$

Now suppose x is in the interior of an interval where b^* is continuous. Observe that

$$\begin{aligned} \sup_{b\in[b_x^*-\delta,b_x^*)} \mathbb{E}\left[V_1 \middle| X_1 = x, W_Z = b\right] & \geq \mathbb{E}\left[V_1 \middle| X_1 = x, b_x^* - \delta \leq W_Z < b_x^*\right] \\ & \geq \inf_{b\in[b_x^*-\delta,b_x^*)} \mathbb{E}\left[V_1 \middle| X_1 = x, W_Z = b\right] \end{aligned}$$

Hence, using the continuity of v_W with respect to b that is established in Lemma 4 we see that $\mathbb{E}\left[V_1 \middle| X_1 = x, b_x^* - \delta \leq W_Z < b_x^*\right]$ approaches $v_W(x, b_x^*)$ as $\delta \searrow 0$. Thus

$\mathbb{E}\left[\left(V_1 - b_x^* + \delta\right)\middle| X_1 = x, b_x^* - \delta \leq W_Z < b_x^*\right]$ approaches $\mathbb{E}\left[V_1 - b_x^*\middle| X_1 = x, W_Z = b_x^*\right]$ as $\delta \searrow 0$. Similarly $\mathbb{E}\left[\left(V_1 - b_x^* - \delta\right)\middle| X_1 = x, b_x^* \leq W_Z < b_x^* + \delta\right]$ approaches the same limit as $\delta \to 0$. Hence if $\mathbb{E}\left[V_1 - b_x^*\middle| X_1 = x, W_Z = b_x^*\right] \neq 0$ we obtain a contradiction either from (15) or from (16). Thus

$$\mathbb{E}\left[V_1 - b_x^*\middle| X_1 = x, W_Z = b_x^*\right] = 0.$$

So we have established $v_W(x, b_x^*) = b_x^*$ provided x is in the interior of an interval where b^* is continuous, i.e. provided that x is not at a break point in the definition of b^*. ■

Proof. (Lemma 6) Consider a potential bid $\widetilde{b}$ (which may differ from the equilibrium bid)made when the signal is x_0. We will show that the change in profit for a small increase in $\widetilde{b}$ has the same sign as $v_W(x_0, \widetilde{b}) - \widetilde{b}$ if $v_W(x_0, \widetilde{b})$ is well defined for $\widetilde{b}$ (and small increases in $\widetilde{b}$). This is enough to show that the optimal choice $b^*(x_0) > \widetilde{b}$ if $v_W(x_0, \widetilde{b}) > \widetilde{b}$.

We start with the case that $v_W(x_0, \widetilde{b}) - \widetilde{b} > 0$. Then

$$\begin{aligned}
& \Pi_1(\widetilde{b} + \delta, x_0) - \Pi_1(\widetilde{b}, x_0) \\
= \; & \mathbb{E}\left[\left(V_1 - W_Z\right) 1_{W_Z < \widetilde{b} + \delta} - \left(V_1 - W_Z\right) 1_{W_Z < \widetilde{b}}\middle| X_1 = x_0\right] \\
= \; & \mathbb{E}\left[\left(V_1 - W_Z\right) 1_{\widetilde{b} < W_Z < \widetilde{b} + \delta}\middle| X_1 = x_0\right] \\
\geq \; & \mathbb{E}\left[\left(V_1 - \widetilde{b} - \delta\right) 1_{\widetilde{b} < W_Z < \widetilde{b} + \delta}\middle| X_1 = x_0\right] \\
= \; & \Pr(\widetilde{b} < W_Z < \widetilde{b} + \delta)\middle| X_1 = x_0))\mathbb{E}\left[V_1 - \widetilde{b} - \delta\middle| X_1 = x_0, \widetilde{b} < W_Z < \widetilde{b} + \delta\right].
\end{aligned}$$

Now as δ approaches zero $\mathbb{E}\left[V_1 - \widetilde{b} - \delta\middle| X_1 = x_0, \widetilde{b} < W_Z < \widetilde{b} + \delta\right]$ approaches $v_W\left(x_0, \widetilde{b}\right) - \widetilde{b} - \delta$ which is positive for δ small enough.

The implication when $v_W(x_0, \widetilde{b}) - \widetilde{b} < 0$ follows similarly, since

$$\mathbb{E}\left[\left(V_1 - W_Z\right) 1_{\widetilde{b} < W_Z < \widetilde{b} + \delta}\middle| X_1 = x_0\right] \leq \mathbb{E}\left[\left(V_1 - \widetilde{b}\right) 1_{\widetilde{b} < W_Z < \widetilde{b} + \delta}\middle| X_1 = x_0\right]$$

and, using the same argument, this has the same sign as $\mathbb{E}\left[V_1 - \widetilde{b}\middle| X_1 = x_0, \widetilde{b} < W_Z < \widetilde{b} + \delta\right]$ which is negative for δ small enough.

Consider two signals $X_1 = x_A \in Q_X$, and $X_1 = x_B \in Q_X$ with x_A in the interior of an interval where b^* is continuous, and with $x_B > x_A$. Then from Lemma 2 and Lemma 5 $v_W(x_B, b^*(x_A)) > v_W(x_A, b^*(x_A)) = b^*(x_A)$. Applying the observation above shows that $b^*(x_B) > b^*(x_A)$. This is sufficient to show that each segment of b^* is strictly increasing. Moreover b^* cannot jump down at some x_A at the end of a segment, since b^* is continuous within a segment and we would have a contradiction from looking at a point in the interior of the segment just below x_A. Thus we deduce that b^* is strictly increasing throughout Q_X. ■

Proof. (Lemma 7) Suppose that there is a jump in the value of b^* at x_0. Write $b^+(\delta) = b^*(x_0 + \delta)$ and $b^-(\delta) = b^*(x_0 - \delta)$ and suppose that $\lim_{\delta \searrow 0} b^+(\delta) = b_A$, and $\lim_{\delta \searrow 0} b^-(\delta) = b_B$, with $b_A > b_B$.

Define the set

$$J_k(x) = \{(x_2, x_3, ...x_n) \ : \ x_2 = x, \ x_3, x_4, ..., x_{k+1} > x, \ x_{k+2}, x_{k+3}, ...x_n < x\}. \tag{17}$$

Now, from Lemma 6, b^* is strictly increasing and using symmetry, conditioning on $W_Z = b^+(\delta)$ when $Z = k$ is equivalent to conditioning on $\mathbf{X}_{-1} \in J_k(x_0 + \delta)$, and hence

$$\mathbb{E}\left[V_1 | \, X_1 = x_0 + \delta, W_Z = b^+(\delta), Z = k\right] = \mathbb{E}\left[V_1 | \, X_1 = x_0 + \delta, \mathbf{X}_{-1} \in J_k(x_0 + \delta)\right].$$

Thus we can apply Lemma 17 to show that

$$\lim_{\delta \searrow 0} \mathbb{E}\left[V_1 | \, X_1 = x_0 + \delta, W_Z = b^+(\delta), Z = k\right] = \mathbb{E}\left[V_1 | \, X_1 = x_0, \mathbf{X}_{-1} \in J_k(x_0)\right].$$

Similarly we can show the same limit for the $x_0 - \delta$ expectation:

$$\lim_{\delta \searrow 0} \mathbb{E}\left[V_1 | \, X_1 = x_0 - \delta, W_Z = b^-(\delta), Z = k\right] = \mathbb{E}\left[V_1 | \, X_1 = x_0, \mathbf{X}_{-1} \in J_k(x_0)\right],$$

and so

$$\lim_{\delta \searrow 0} \mathbb{E}\left[V_1 | \, X_1 = x_0 + \delta, W_Z = b^+(\delta), Z = k\right] = \lim_{\delta \searrow 0} \mathbb{E}\left[V_1 | \, X_1 = x_0 - \delta, W_Z = b^-(\delta), Z = k\right].$$

Now from independence of Z and since b^* is strictly increasing and symmetric,

$$\begin{aligned} & v_W\left(x, b^*(x)\right) \\ = \; & \frac{\sum_{k \in K_Z} \Pr(Z = k) \Pr(\, X_1 = x, \mathbf{X}_{-1} \in J_k(x) | \, Z = k) \mathbb{E}[V_1 | \, X_1 = x, W_Z = b, Z = k]}{\sum_{k \in K_Z} \Pr(Z = k) \Pr(\, X_1 = x, \mathbf{X}_{-1} \in J_k(x) | \, Z = k)}. \end{aligned}$$

Since $\Pr(\, X_1 = x_0 - \delta, \mathbf{X}_{-1} \in J_k(x_0 - \delta) | \, Z = k)$ and $\Pr(\, X_1 = x_0 + \delta, \mathbf{X}_{-1} \in J_k(x_0 + \delta) | \, Z = k)$ both have the limit $\Pr(\, X_1 = x_0, \mathbf{X}_{-1} \in J_k(x_0) | \, Z = k)$ as $\delta \searrow 0$, we may deduce that

$$\lim_{\delta \searrow 0} v_W(x_0 + \delta, b^+(\delta)) = \lim_{\delta \searrow 0} v_W(x_0 - \delta, b^-(\delta))$$

Since $v_W(x_0 + \delta, b^*(x_0 + \delta)) = b^*(x_0 + \delta)$ from Lemma 5 we have $\lim_{\delta \searrow 0} b^*(x_0 + \delta)) = \lim_{\delta \searrow 0} b^*(x_0 - \delta))$ contradicting our initial assumption, which establishes the result we require. ■

Proof. (Lemma 8) If there is more than one equilibrium, say b_1^* and b_2^*, then there is some signal $X_1 = x_0 \in Q_X$ where $b_1^*(x_0) \neq b_2^*(x_0)$. We let $W_Z(b_i^*)$ be the Zth highest bid from the competitors under b_i^*. Then from Lemma 5

$$b_i^*(x_0) = \mathbb{E}\left[V_1 | \, X_1 = x_0, W_Z(b_i^*) = b_i^*(x_0)\right], \, i = 1, 2. \tag{18}$$

Using Lemma 6 b_i^* are strictly increasing and so for equilibrium b_i^*, $i = 1, 2$, we have

$$\mathbb{E}[V_1 | \, X_1 = x, W_Z(b_i^*) = b, Z = k] = \mathbb{E}\left[V_1 | \, X_1 = x, \mathbf{X}_{-1} \in J_k(x), Z = k\right],$$

where $J_k(x)$ is defined in (17). Hence under the equilibrium b_1^*,

$$\mathbb{E}\left[V_1 | \, X_1 = x_0, W_Z(b_1^*) = b_1^*(x_0), Z = k\right] = \mathbb{E}\left[V_1 | \, X_1 = x_0, \mathbf{X}_{-1} \in J_k(x_0), Z = k\right].$$

Since the expression on the right-hand side is independent of the bids b_1^* , it follows that

$$\mathbb{E}\left[V_1|\, X_1 = x_0, W_Z(b_1^*) = b_1^*(x_0), Z = k\right] = \mathbb{E}\left[V_1|\, X_1 = x_0, W_Z(b_2^*) = b_2^*(x_0), Z = k\right]$$

for each value of k. Moreover $\Pr(X_1 = x, \mathbf{X}_{-1} \in J_k(x)|\, Z = k)$ is also independent of the bids and so, using the expression

$$\begin{aligned} & v_W\left(x, b^*(x)\right) \\ = & \frac{\sum_{k\in K_Z} \Pr(Z=k)\Pr(X_1 = x, \mathbf{X}_{-1} \in J_k(x)|\, Z = k)\mathbb{E}[V_1|\, X_1 = x, W_Z = b^*(x), Z = k]}{\sum_{k\in K_Z} \Pr(Z=k)\Pr(X_1 = x, \mathbf{X}_{-1} \in J_k(x)|\, Z = k)}, \end{aligned}$$

we can deduce that

$$\mathbb{E}\left[V_1|\, X_1 = x_0, W_Z(b_1^*) = b_1^*(x_0)\right] = \mathbb{E}\left[V_1|\, X_1 = x_0, W_Z(b_2^*) = b_2^*(x_0)\right]$$

and so from (18) we obtain $b_1^*(x_0) = b_2^*(x_0)$, a contradiction. ■

Proof. (Proposition 3) It follows from Lemma 5 and (2) that $b^*(x) = v(x,x)$ is the necessary first-order condition for symmetric and strictly increasing bids. It follows from Lemma 1 and Theorem 5 in Milgrom & Weber (1982) that $v(x,x)$ is strictly increasing with respect to x, which confirms our assumption on $b^*(x)$. It also follows from the first-order condition that there is only one equilibrium candidate with symmetric and strictly increasing bids.

The proof of sufficiency corresponds to the proof of Theorem 6 in Milgrom & Weber (1982). We will show that $b^*(x)$ is an optimal response when all other players use the same strategy. Using that the bid function is strictly monotonic, it follows that bidder 1's conditional expected profit when it bids b is:

$$\begin{aligned} & \mathbb{E}\left[\left(V_1 - b^*\left(Y_Z\right)\right) 1_{b^*(Y_Z)<b} \right| X_1 = x\big] \\ = & \mathbb{E}\left[\left(v\left(X_1, Y_Z\right) - v\left(Y_Z, Y_Z\right)\right) 1_{b^*(Y_Z)<b} \right| X_1 = x\big]. \end{aligned}$$

Due to the properties of v, the difference $v\left(X_1, Y_Z\right) - v\left(Y_Z, Y_Z\right)$ is non-negative for $X_1 \geq Y_Z$ and non-positive for $X_1 \leq Y_Z$. Hence, it follows that the expression above is maximized when the indicator function is one for $X_1 \geq Y_Z$, and zero for $X_1 < Y_Z$. This is the case when bidder 1 bids $b^*\left(X_1\right)$. ■

Proof. (Lemma 9) The proof is inspired by the proof of Theorem 8 in Milgrom & Weber (1982). One difference is that we replace Y_1 by Y_Z. Another is that we prove the inequality for each realization of X_1 (We need this when we rank revenues in Corollary 1). First, note that

$$v\left(x, y\right) = \mathbb{E}\left[\hat{v}\left(x, y; X_0\right)\right].$$

Moreover, it follows from Theorem 5 in Milgrom & Weber (1982) that:

$$v\left(u, u\right) = \mathbb{E}\left[\hat{v}\left(u, u; X_0\right)|\, X_1 = u, Y_Z = u\right] \leq \mathbb{E}\left[\hat{v}\left(u, u; X_0\right)|\, X_1 = x, Y_Z = u\right], \tag{19}$$

if $x \geq u$. The inequality arises because if the expected value is conditioned on a higher X_1 signal, then this tends to give a higher X_0 signal, which in its turn increases the value of $\hat{v}$. This effect disappears if X_0 is independent of $X_1, \ldots, X_n$ and Z. In this special case, the inequality above becomes an equality. This is also true for the inequalities in the rest of the proof.

If we let u be a random variable and if we condition both sides of (19) on $X_1 = x$, and $u < x$ then we get (in expectation):

$$\mathbb{E}\left[v\left(Y_Z, Y_Z\right) \mid X_1 = x, Y_Z < x\right] \leq \mathbb{E}\left[\hat{v}\left(Y_Z, Y_Z; X_0\right) \mid X_1 = x, Y_Z < x\right]. \tag{20}$$

Now, using Proposition 3 and the inequality above, we get:

$$\begin{aligned} P^{\mathbb{N}}(x) &= \mathbb{E}\left[b\left(Y_Z\right) \mid X_1 = x, Y_Z < x\right] \\ &= \mathbb{E}\left[v\left(Y_Z, Y_Z\right) \mid X_1 = x, Y_Z < x\right] \\ &\leq \mathbb{E}\left[\hat{v}\left(Y_Z, Y_Z; X_0\right) \mid X_1 = x, Y_Z < x\right] \\ &= \mathbb{E}\left[\hat{b}^*\left(Y_Z, X_0\right) \middle| X_1 = x, Y_Z < x\right] \\ &= P^{\mathbb{I}}(x). \end{aligned}$$

■

Proof. (**Proposition 4)** Using the same argument as in the proof of Theorem 9 in Milgrom & Weber (1982), it can be shown that if we condition the probability distribution on X_0', then $S_1, \ldots, S_m, X_0, X_1, \ldots, X_n$, Y_Z are still affiliated. Thus, we can conclude from Corollary 1 that the expected price and expected revenue will weakly increase if X_0 is disclosed, in addition to X_0'. But as argued in Milgrom & Weber (1982) disclosing X_0 is as informative, and has the same effect on revenue and price, as to disclose both X_0 and X_0'. This proves our statement. ■

Proofs of Section 5

Proof. (Lemma 10) If the considered bidder observes the private signal x and acts as if observing $\widetilde{x}$, then its expected payoff is given by:

$$U(\widetilde{x}, x) - K(\widetilde{x}, x).$$

In equilibrium, we have that it is optimal for the bidder to choose $\widetilde{x} = x$, so

$$K_{\widetilde{x}}(x, x) = U_{\widetilde{x}}(x, x), \tag{21}$$

where $U_{\widetilde{x}}(x, x)$, and consequently also $K_{\widetilde{x}}(x, x)$, is independent of the auction design. We have from (5) and (21) that:

$$U_{\widetilde{x}}(x, x) = K_{\widetilde{x}}(x, x) = J_{\widetilde{x}}(x, x) \Pr\left(Y_Z \leq x \mid x\right) + J(x, x) \left. \frac{d \Pr\left(Y_Z \leq \widetilde{x} \mid x\right)}{d\widetilde{x}} \right|_{\widetilde{x}=x} \tag{22}$$

$$J_{\widetilde{x}}(x, x) = \frac{U_{\widetilde{x}}(x, x) - J(x, x) \left. \frac{d \Pr\left(Y_Z \leq \widetilde{x} \mid x\right)}{d\widetilde{x}} \right|_{\widetilde{x}=x}}{\Pr\left(Y_Z \leq x \mid x\right)}, \tag{23}$$

where $\Pr\left(Y_Z \leq x \mid x\right) > 0$ for all $x \in Q_X$. Moreover, we have

$$\frac{dJ(x, x)}{dx} = J_{\widetilde{x}}(x, x) + J_x(x, x). \tag{24}$$

Hence,

$$\begin{aligned}
\frac{d\left(J^{\mathbb{I}}(x,x)-J^{\mathbb{III}}(x,x)\right)}{dx} &= J^{\mathbb{I}}_{\widetilde{x}}(x,x)+J^{\mathbb{I}}_{x}(x,x)-J^{\mathbb{III}}_{\widetilde{x}}(x,x)-J^{\mathbb{III}}_{x}(x,x). \\
&= \frac{\left(J^{\mathbb{III}}(x,x)-J^{\mathbb{I}}(x,x)\right)\left.\frac{d\Pr(Y_Z\le\widetilde{x}|x)}{d\widetilde{x}}\right|_{\widetilde{x}=x}}{\Pr\left(Y_Z\le x|\,x\right)} \\
&\quad +J^{\mathbb{I}}_{x}(x,x)-J^{\mathbb{III}}_{x}(x,x).
\end{aligned}$$

From the inequality $J^{\mathbb{I}}_{x}(x,x)\ge J^{\mathbb{III}}_{x}(x,x)$ we now get

$$\begin{aligned}
\frac{d\left(J^{\mathbb{I}}(x,x)-J^{\mathbb{III}}(x,x)\right)}{dx} &\ge \frac{\left(J^{\mathbb{III}}(x,x)-J^{\mathbb{I}}(x,x)\right)\left.\frac{d\Pr(Y_Z\le\widetilde{x}|x)}{d\widetilde{x}}\right|_{\widetilde{x}=x}}{\Pr\left(Y_Z\le x|\,x\right)}. \\
\frac{d\left(J^{\mathbb{III}}(x,x)-J^{\mathbb{I}}(x,x)\right)}{dx} &\le -\frac{\left(J^{\mathbb{III}}(x,x)-J^{\mathbb{I}}(x,x)\right)\left.\frac{d\Pr(Y_Z\le\widetilde{x}|x)}{d\widetilde{x}}\right|_{\widetilde{x}=x}}{\Pr\left(Y_Z\le x|\,x\right)}.
\end{aligned}$$

Thus, it follows from Grönwall's lemma that:

$$J^{\mathbb{III}}(x,x)-J^{\mathbb{I}}(x,x)\le\left(J^{\mathbb{III}}(a,a)-J^{\mathbb{I}}(a,a)\right)\exp\left(-\int_a^x\frac{\left.\frac{d\Pr(Y_Z\le\widetilde{x}|v)}{d\widetilde{x}}\right|_{\widetilde{x}=v}}{\Pr\left(Y_Z\le v|\,v\right)}dv\right), \tag{25}$$

for $a,x\in Q_X$. Note that

$$0\le\exp\left(-\int_a^x\frac{\left.\frac{d\Pr(Y_Z\le\widetilde{x}|v)}{d\widetilde{x}}\right|_{\widetilde{x}=v}}{\Pr\left(Y_Z\le v|\,v\right)}dv\right)\le 1 \tag{26}$$

for $a,x\in Q_X$. This follows from observing that $\left.\frac{d\Pr(Y_Z\le\widetilde{x}|v)}{d\widetilde{x}}\right|_{\widetilde{x}=v}$, and hence the integrand, is non-negative. Moreover, we have by assumption that $\lim_{x\searrow a_L}J^{\mathbb{III}}(x,x)\le\lim_{x\searrow a_L}J^{\mathbb{I}}(x,x)$. Hence, by choosing an a value sufficiently close to a_L, we can use (25) and (26) to find a contradiction for any $x\in Q_X$ such that $J^{\mathbb{III}}(x,x)>J^{\mathbb{I}}(x,x)$. Thus we can conclude that $J^{\mathbb{I}}(x,x)\ge J^{\mathbb{III}}(x,x)$ for $x\in Q_X$. ■

Proof. (Lemma 11) It follows directly from Lemma 10 that $J^{\mathbb{I}}(x,x)\ge J^{\mathbb{III}}(x,x)$ for $x\in Q_X$. Next, we want to show that this implies that $K^{\mathbb{I}}(x,x)\ge K^{\mathbb{III}}(x,x)$. From (5) we can deduce that

$$\begin{aligned}
K_x(x,x) &= J_x(x,x)\Pr\left(Y_Z\le x|\,x\right)+J(x,x)\left.\frac{d\Pr\left(Y_Z\le\widetilde{x}|\,x\right)}{dx}\right|_{\widetilde{x}=x} && (27)\\
&= J_x(x,x)\Pr\left(Y_Z\le x|\,x\right)+\frac{K(x,x)}{\Pr\left(Y_Z\le x|\,x\right)}\left.\frac{d\Pr\left(Y_Z\le\widetilde{x}|\,x\right)}{dx}\right|_{\widetilde{x}=x}.
\end{aligned}$$

Moreover, we have from (21)

$$\frac{dK(x,x)}{dx}=K_{\widetilde{x}}(x,x)+K_x(x,x)=U_{\widetilde{x}}(x,x)+K_x(x,x). \tag{28}$$

Hence we have from (27) and the assumption $J_x^{\mathrm{I}}(x,x) - J_x^{\mathrm{III}}(x,x) \geq 0$ for $x \in Q_X$ that,

$$\begin{aligned}
&\frac{d\left(K^{\mathrm{I}}(x,x) - K^{\mathrm{III}}(x,x)\right)}{dx} \\
= \; & \left(J_x^{\mathrm{I}}(x,x) - J_x^{\mathrm{III}}(x,x)\right) \Pr\left(Y_Z \leq x \middle| x\right) + \left(K^{\mathrm{I}}(x,x) - K^{\mathrm{III}}(x,x)\right) \frac{\left.\frac{d\Pr(Y_Z \leq \widetilde{x}|x)}{dx}\right|_{\widetilde{x}=x}}{\Pr\left(Y_Z \leq x \middle| x\right)} \\
\geq \; & \left(K^{\mathrm{I}}(x,x) - K^{\mathrm{III}}(x,x)\right) \frac{\left.\frac{d\Pr(Y_Z \leq \widetilde{x}|x)}{dx}\right|_{\widetilde{x}=x}}{\Pr\left(Y_Z \leq x \middle| x\right)}.
\end{aligned}$$

The inequality can be written as follows:

$$\frac{d\left(K^{\mathrm{III}}(x,x) - K^{\mathrm{I}}(x,x)\right)}{dx} \leq \left(K^{\mathrm{III}}(x,x) - K^{\mathrm{I}}(x,x)\right) \frac{\left.\frac{d\Pr(Y_Z \leq \widetilde{x}|x)}{dx}\right|_{\widetilde{x}=x}}{\Pr\left(Y_Z \leq x \middle| x\right)}.$$

It now follows from Grönwall's lemma that:

$$K^{\mathrm{III}}(x,x) - K^{\mathrm{I}}(x,x) \leq \left(K^{\mathrm{III}}(a,a) - K^{\mathrm{I}}(a,a)\right) \exp\left(\int_a^x \frac{\left.\frac{d\Pr(Y_Z \leq \widetilde{x}|v)}{dv}\right|_{\widetilde{x}=v}}{\Pr\left(Y_Z \leq v \middle| v\right)} dv\right), \tag{29}$$

for $a, x \in Q_X$. Since $\left.\frac{d\Pr(Y_Z \leq \widetilde{x}|v)}{dv}\right|$ is non-positive for affiliated signals (Assumption 5), we can deduce that

$$0 \leq \exp\left(\int_a^x \frac{\left.\frac{d\Pr(Y_Z \leq \widetilde{x}|v)}{dv}\right|_{\widetilde{x}=v}}{\Pr\left(Y_Z \leq v \middle| v\right)} dv\right) \leq 1 \tag{30}$$

for $a, x \in Q_X$. Moreover, we have by assumption that $\lim_{x \searrow a_L} K^{\mathrm{I}}(x,x) = \lim_{x \searrow a_L} K^{\mathrm{III}}(x,x) = 0$. Using a similar argument as in the proof of Lemma 10, it can be shown that $K^{\mathrm{I}}(x,x) \geq K^{\mathrm{III}}(x,x)$ for $x \in Q_X$. ■

9.1 Proofs of Section 6

Lemma 18 $\Pr\left(Y_Z \leq \widetilde{x} \middle| X_1 = x\right)$, *$U(\widetilde{x}, x)$ and $v(x, \widetilde{x})$ are differentiable with respect to $\widetilde{x}$ and x, if $\widetilde{x}, x \in Q_X$, and Assumption 6 is satisfied.*

Proof. From Milgrom & Weber (1982), we know that the joint density for $\boldsymbol{S}, X_1, \boldsymbol{Y}$ is given by:

$$\widetilde{f}_k(\boldsymbol{S}, X_1, \boldsymbol{Y}) = (n-1)! f_k(\boldsymbol{S}, X_1, \boldsymbol{Y}) 1_{\{y_1 \geq .. \geq y_{n-1}\}}. \tag{31}$$

It follows from Assumption 6 that this density is differentiable with respect to X_1 and $\boldsymbol{Y}$, if $X_1 \in Q_X$, $\boldsymbol{Y} \in Q_Y$, $\boldsymbol{S} \in Q_S$, and $k \in K_Z$, where

$$Q_Y = \left\{\mathbf{y} \in Q_X \times \ldots \times Q_X \left| y_i \leq y_{i+1} \text{ for } i = 1 \ldots n-1\right.\right\}.$$

We also define

$$
\begin{aligned}
Q_Y(k, \overline{y}) &= \{\mathbf{y} \in Q_X \times \ldots \times Q_X \,|y_i \leq y_{i+1} \text{ for } i = 1 \ldots n-1 \text{ and } y_k \leq \overline{y}\} \\
\overline{Q}_Y(k, \overline{y}) &= \{\mathbf{y} \in Q_X \times \ldots \times Q_X \,|y_i \leq y_{i+1} \text{ for } i = 1 \ldots n-1 \text{ and } y_k = \overline{y}\}
\end{aligned}
$$

Using these definitions and the joint density in (31), we get

$$
\begin{aligned}
& \Pr\left(Y_Z \leq \widetilde{x} \middle| X_1 = x\right) \\
= \; & \frac{\sum_{k\in K_Z} \Pr(Z=k) \int_{\boldsymbol{S}\in Q_S, \boldsymbol{Y}\in Q_Y(k,\widetilde{x})} \widetilde{f}_k(\boldsymbol{S}, x, \boldsymbol{Y})\, d\boldsymbol{S} d\boldsymbol{Y}}{\sum_{k\in K_Z} \Pr(Z=k) \int_{\boldsymbol{S}\in Q_S, \boldsymbol{Y}\in Q_Y} \widetilde{f}_k(\boldsymbol{S}, x, \boldsymbol{Y})\, d\boldsymbol{S} d\boldsymbol{Y}},
\end{aligned}
$$

which is differentiable with respect to $\widetilde{x} \in Q_X$ and $x \in Q_X$. This is the first result of the lemma. Moreover,

$$
\begin{aligned}
U(\widetilde{x}, x) &= \mathbb{E}\left[V_1 \middle| Y_Z \leq \widetilde{x} \text{ and } X_1 = x\right] \\
&= \frac{\sum_{k\in K_Z} \Pr(Z=k) \int_{\boldsymbol{S}\in Q_S, \boldsymbol{Y}\in Q_Y(k,\widetilde{x})} u(\boldsymbol{S}, x, \boldsymbol{Y})\, \widetilde{f}_k(\boldsymbol{S}, x, \boldsymbol{Y})\, d\boldsymbol{S} d\boldsymbol{Y}}{\sum_{k\in K_Z} \Pr(Z=k) \int_{\boldsymbol{S}\in Q_S, \boldsymbol{Y}\in Q_Y(k,\widetilde{x})} \widetilde{f}_k(\boldsymbol{S}, x, \boldsymbol{Y})\, d\boldsymbol{S} d\boldsymbol{Y}},
\end{aligned}
$$

is differentiable with respect to $\widetilde{x}$ and x. We also have

$$
\begin{aligned}
v(x, y) &= \mathbb{E}\left[V_1 \middle| X_1 = x, Y_Z = y\right] \\
&= \frac{\sum_{k\in K_Z} \Pr(Z=k) \int_{\boldsymbol{S}\in Q_S, \boldsymbol{Y}\in \overline{Q}_Y(k,y)} u(\boldsymbol{S}, x, \boldsymbol{Y})\, \widetilde{f}_k(\boldsymbol{S}, x, \boldsymbol{Y})\, d\boldsymbol{S} d\boldsymbol{Y}}{\sum_{k\in K_Z} \Pr(Z=k) \int_{\boldsymbol{S}\in Q_S, \boldsymbol{Y}\in \overline{Q}_Y(k,y)} \widetilde{f}_k(\boldsymbol{S}, x, \boldsymbol{Y})\, d\boldsymbol{S} d\boldsymbol{Y}},
\end{aligned}
$$

which is differentiable with respect to x and y. ■

Proof. (Proposition 5) Let $\tau(b) = b^{*-1}(b)$. Bidder 1 will choose its bid optimally. Thus we differentiate $\Pi(b, x)$ in (8) with respect to b. Leibniz' rule gives us:

$$
\begin{aligned}
\frac{\partial \Pi(b, x)}{\partial b} &= \tau'(b)\left(v(x, \tau(b)) - b\right) f_{Y_Z}\left(\tau(b) \middle| x\right) \\
&\quad - F_{Y_Z}\left(\tau(b) \middle| x\right).
\end{aligned} \tag{32}
$$

Hence, the symmetric equilibrium candidate $b^*(x)$, where $\tau(b) = x$ and $b = b^*(x)$, can be found from the following differential equation

$$
0 = \frac{1}{b^{*\prime}(x)}\left(v(x, x) - b^*(x)\right) f_{Y_Z}\left(x \middle| x\right) - F_{Y_Z}\left(x \middle| x\right), \tag{33}
$$

which can be simplified to the differential equation:

$$
b^{*\prime}(x) = \left(v(x, x) - b^*(x)\right) \frac{f_{Y_Z}\left(x \middle| x\right)}{F_{Y_Z}\left(x \middle| x\right)}. \tag{34}
$$

The solution to the ODE can be found in Milgrom & Weber (1982), and is given in (9). The solution implies that $v(x, x) > b^*(x)$. We have that $F_{Y_Z}(x|x) > 0$ for $x \in Q_X$. Hence, it follows from (34) that $b^{*\prime}(x) > 0$ and finite. Moreover, it follows that $b^*(x)$ is continuous for a given $x \in Q_X$. Next, we want to verify that $b^*(x)$ is the best response of bidder 1. It follows from (32) that:

$$\frac{\partial \Pi(b,x)}{\partial b} = f_{Y_Z}(\tau(b)|\,x)\,\tau'(b)\left(v(x,\tau(b)) - b - \frac{1}{\tau'(b)}\frac{F_{Y_Z}(\tau(b)|\,x)}{f_{Y_Z}(\tau(b)|\,x)}\right). \quad (35)$$

Since we assume Y_Z is affiliated, it follows (see Lemma 1 in Milgrom & Weber (1982)) that $\frac{F_{Y_Z}(\tau(b)|x)}{f_{Y_Z}(\tau(b)|x)}$ is non-increasing with respect to x. Moreover, $v(x,\tau(b))$ is non-decreasing with respect to x. We have $\frac{\partial \Pi(b,x)}{\partial b} = 0$ for $x = \tau(b)$. Thus, it follows from (35) that $\frac{\partial \Pi(b,x)}{\partial b} \geq 0$ for $x \geq \tau(b)$ and $\frac{\partial \Pi(b,x)}{\partial b} \leq 0$ for $x \leq \tau(b)$. Equivalently, $\frac{\partial \Pi(b,x)}{\partial b} \geq 0$ for $b^*(x) \geq b$ and $\frac{\partial \Pi(b,x)}{\partial b} \leq 0$ for $b^*(x) \leq b$. Thus we can conclude that $b^*(x)$ is the best response of bidder 1. ■

Lemma 19 *For a pay-as-bid auction, $J^{\mathbb{N}}(\widetilde{x},x)$, $J^{\mathbb{I}}(\widetilde{x},x)$, $K^{\mathbb{N}}(\widetilde{x},x)$ and $K^{\mathbb{I}}(\widetilde{x},x)$ are differentiable with respect to $\widetilde{x}$ and x if $\widetilde{x}, x \in Q_X$ and Assumption 6 is satisfied.*

Proof. Conditional on acceptance, $J(\widetilde{x},x)$ is the expected payment when the bidder observes $x \in Q_X$ and bids as if observing $\widetilde{x} \in Q_X$. We have from Proposition 5 that

$$\begin{aligned} J^{\mathbb{N}}(\widetilde{x},x) &= b^*(\widetilde{x}) = v(\widetilde{x},\widetilde{x}) - \int_{a_L}^{\widetilde{x}} L(\alpha|\,\widetilde{x})\,dt(\alpha), & (36)\\ L(\alpha|\,\widetilde{x}) &= \exp\left(-\int_{\alpha}^{\widetilde{x}} \frac{f_{Y_Z}(s|\,s)}{F_{Y_Z}(s|\,s)}ds\right) & (37)\\ t(\alpha) &= v(\alpha,\alpha), & (38)\end{aligned}$$

which is differentiable with respect to x and $\widetilde{x}$, because we know from Lemma 18 that $v(x,y)$ is differentiable with respect to both x and y. Similarly, in the $\mathbb{I}$ auction the optimal bid of bidder 1 observing signal x and X_0 is

$$\begin{aligned} \widehat{b}(x,X_0) &= \widehat{v}(x,x,X_0) - \int_{a_L}^{x} \widehat{L}(\alpha|\,x,X_0)\,d\widehat{t}(\alpha,X_0) & (39)\\ \widehat{L}(\alpha|\,x,X_0) &= \exp\left(-\int_{\alpha}^{x} \frac{f_{Y_Z}(s|\,s,X_0)}{F_{Y_Z}(s|\,s,X_0)}ds\right) & (40)\\ \widehat{t}(\alpha,X_0) &= \widehat{v}(\alpha,\alpha,X_0), & (41)\end{aligned}$$

which is differentiable with respect to x. Differentiability of $J^{\mathbb{I}}(\widetilde{x},x)$, $K^{N}(\widetilde{x},x)$ and $K^{I}(\widetilde{x},x)$ can be proved in a similar way to the proof of Lemma 18. ■

Proof. (Proposition 6) When the information is not disclosed, we have, similar to the proof of Lemma 19, that

$$\begin{aligned} J^{\mathbb{N}}(\widetilde{x},x) &= b^{\mathbb{N}}(\widetilde{x}), \\ J^{\mathbb{N}}_x &= 0, \\ \lim_{x\searrow a_L} J^{\mathbb{N}}(x,x) &= \lim_{x\searrow a_L} v(x,x). \end{aligned}$$

In the case, when the auctioneer discloses X_0, we have that there is a symmetric BNE where each bidder $i \in \{1, \dots, n\}$ makes the bid $b^{\mathbb{I}}(x; X_0)$. In this case we get:

$$\begin{aligned} J^{\mathbb{I}}(\widetilde{x}, x) &= \mathbb{E}\left[b^{\mathbb{I}}(\widetilde{x}, X_0) \middle| X_1 = x \right], \\ \lim_{x \searrow a_L} J^{\mathbb{I}}(x, x) &= \lim_{x \searrow a_L} \mathbb{E}\left[b^{\mathbb{I}}(x, X_0) \middle| X_1 = x \right] \\ &= \lim_{x \searrow a_L} \mathbb{E}\left[\mathbb{E}\left[V_1 \middle| X_1 = x, Y_Z = x, X_0 = x_0 \right] \right] \\ &= \lim_{x \searrow a_L} v(x, x). \end{aligned}$$

We have that $b^{\mathbb{I}}(x; X_0)$ is non-decreasing in x. Using the same argument as in the proof of Theorem 16 in Milgrom & Weber (1982), it can also be shown that $b^{\mathbb{I}}(x; X_0)$ is non-decreasing in X_0. Hence, it follows from Theorem 5 in Milgrom & Weber (1982) that

$$J_x^{\mathbb{I}} \geq 0.$$

In the special case where X_0 is independent of $X_1, \dots, X_n$ and Z we have $J_x^{\mathbb{I}} = 0$. The statement now follows from the above and Lemma 11. ∎

Lemma 20 *For a uniform-price auction, $J(\widetilde{x}, x)$ and $K(\widetilde{x}, x)$ are differentiable with respect to $\widetilde{x}$ and x if $\widetilde{x}, x \in Q_X$ and Assumption 6 is satisfied.*

Proof. For the uniform-price auction we have the bid function

$$b(x) = v(x, x),$$

which is differentiable with respect to x, because we know from Lemma 18 that $v(x, y)$ is differentiable with respect to both x and y. Differentiability of $J(\widetilde{x}, x)$ and $K(\widetilde{x}, x)$ can be proved in a similar way to the proof of Lemma 18. ∎

Proof. (Proposition 8) We use the superscripts $\mathbb{U}$ and $\mathbb{P}$ to denote a uniform-price and pay-as-bid auction, respectively. Conditional on acceptance, $J(\widetilde{x}, x)$ is the expected payment when the bidder observes $x \in Q_X$ and bids as if observing $\widetilde{x} \in Q_X$. In the uniform-price auction the price is set by the bid $b^{\mathbb{U}}(Y_Z)$ if bidder 1's bid is accepted. Hence, it follows from Proposition 3 that

$$J^{\mathbb{U}}(\widetilde{x}, x) = \mathbb{E}\left[b(Y_Z) \middle| X_1 = x, Y_Z \leq \widetilde{x} \right] = \mathbb{E}\left[v(Y_Z, Y_Z) \middle| X_1 = x, Y_Z \leq \widetilde{x} \right]. \tag{42}$$

We know that v is non-decreasing in its arguments. Hence, it follows from Theorem 5 in Milgrom & Weber (1982) that

$$J_x^{\mathbb{U}} \geq 0.$$

In the special case where $X_1, \dots, X_n, Z$ are all independent it follows that $J_x^{\mathbb{U}} = 0$. Moreover, we have

$$\lim_{x \searrow a_L} J^{\mathbb{U}}(x, x) = \lim_{x \searrow a_L} \mathbb{E}\left[v(Y_Z, Y_Z) \middle| X_1 = x, Y_Z = x \right] = \lim_{x \searrow a_L} v(x, x).$$

For the pay-as-bid auction, we have

$$J^{\mathbb{P}}(\widetilde{x}, x) = b^{\mathbb{P}}(\widetilde{x}).$$

Hence,

$$J_x^{\mathbb{P}} = 0.$$

Moreover, we have from Proposition 5 that

$$\lim_{x \searrow a_L} J^{\mathbb{P}}(x, x) = \lim_{x \searrow a_L} v(x, x).$$

The statement now follows from the above and Lemma 11. ■

Proofs and technical lemmas of Section 7

Proof. (**Lemma 12**) Because applying reflection twice brings us back to the original auction we only need to prove this implication in one direction. Suppose that we have an equilibrium in auction A. Consider a realisation in auction A in which the firm values are $(v_1, v_2, ..., v_n)$ and the quantity is Z_A. Consider firm 1 with value v_1. Let W_A be the Z_Ath highest bid amongst the other bids in this equilibrium. If firm 1 makes the bid $b(v_1)$ for this signal then the profit for firm 1 conditional on this signal is $v_1 - W_A$ when $W_A < b(v_1)$ and $p_A(v_1 - W_A)$ when $b(v_1) = W_A$ (where $p_A < 1$ is the probability of being accepted in case of multiple bids at the clearing price) and zero otherwise. Recall that in case of ties, we have assumed that acceptance is determined randomly, such that each bid at the clearing price has the same chance of being accepted.

Now we consider the profit made by firm 1 in the reflected auction at this realisation. Firm 1 has value $-v_1$ in the reflected auction and bids $-b(v_1)$. Let W_B be the Z_Bth highest bid amongst firms $j \neq 1$ in the reflected auction, where $Z_B = n - Z_A$. Because the order of bids is reversed, if firm j bids at W_A (and has the Z_Ath highest bid) in auction A, this firm will bid at W_B and have the the Z_Bth highest bid in the reflected auction. Hence $W_B = -W_A$. A bid strictly above W_A in auction A is accepted, but after reversing the order, the bid is strictly below W_B (and rejected) in Auction B, and vice versa.If firm 1 bids at W_A it will also bid at W_B in the reflected auction. We will show that the acceptance probability for a rationed bid at W_A in Auction A becomes the rejection probability for a rationed bid at W_B in Auction B. Suppose $b(v_1) = W_A$ in auction A and that there are k bids at W_A of which ℓ are accepted, so that $p_A = \ell/k$. Then in the reflected auction there will be $k - \ell$ accepted from this set, meaning that there will be a probability of acceptance for firm 1 of $(k - \ell)/k = 1 - p_A$. Hence in the reflected auction the profit for firm 1 with value $-v_1$ is: $-v_1 - W_B = -v_1 + W_A$ when $W_B < -b(v_1)$ and $(1 - p_A)(-v_1 - W_B)$ when $-b(v_1) = W_B$ and zero otherwise.

The expected profit for firm 1 in auction A conditional on the signal v_1 is

$$\mathbb{E}[\Pi_1 \mid x_1 = v_1] = \mathbb{E}[(v_1 - W_A)I_{W_A < b(v_1)} + p_A(v_1 - W_A)I_{W_A = b(v_1)} \mid x_1 = v_1].$$

At an equilibrium this is maximized by the bid $b(v_1)$.

The expected profit for firm 1 in the reflected auction is

$$\begin{aligned}
&= \mathbb{E}[(-v_1 - W_B)I_{W_B<-b(v_1)} + (1-p_A)(-v_1 - W_B)I_{W_B=-b(v_1)} \mid x_1 = -v_1] \\
&= \mathbb{E}[(-v_1 + W_A)I_{W_A>b(v_1)} + (1-p_A)(-v_1 + W_A)I_{W_A=b(v_1)} \mid x_1 = v_1] \\
&= \mathbb{E}[\Pi_1 \mid x_1 = v_1] - \mathbb{E}[v_1 - W_A \mid x_1 = v_1].
\end{aligned}$$

The term $\mathbb{E}[v_1 - W_A| \, x_1 = v_1]$ is conditional on the signal v_1 but is independent of firm 1's bid. Thus the bids in the reflected auction maximize expected profit and they must be an equilibrium in the reflected auction. ■

Proof. (Lemma 13) Ex-post optimality implies that a firm's bid is not accepted at a price above its value or rejected when the price is below its value. Hence, a bid at value is always ex-post optimal. First consider the case where a firm's equilibrium bid is accepted at a price above its value. Hence, the firm's bid is above its private value and one or more competitor's bids are price setting and strictly between these values or equal to the firm's bid. It follows from our assumption on regular bids that this occurs with a positive probability, with a possible exception for ties.[40] The firm can avoid unprofitable outcomes by reducing its bid to the firm's valuation. The firm's bid is never price setting when its bid is accepted in a uniform-price auction. Hence, such a change will not influence the payoff from profitable outcomes where the original bid was accepted at a marginal price below the valuation. The expected profit would be improved by such a change, contradicting the assumption that this is an equilibrium.

Next consider the case where the firm's equilibrium bid is rejected at a price below its value with a positive probability. Hence, the firm's bid is below the value of the firm and a set of competitor's bids are accepted and strictly between these values or equal to the firm's bid. Hence, the firm can increase its payoff by increasing its bid to its valuation. ■

Proof. (Proposition 9) First we note that $\max\{\overline{V}_i : i \notin \Omega\} > \min\{\underline{V}_i : i \in \Omega\}$, because Assumption B and our definitions imply that $\min\{\overline{V}_i\} > V_L' \geq V_L \geq \min\{\underline{V}_i : i \in \Omega\}$. Hence, bids from high-bidding firms are always accepted and bids from the low-bidding firms are always rejected. Low bidding firms not in Ω have zero profit, but cannot improve unless they are accepted with a positive probability. This would require them to bid at a higher value than $\max\{\overline{V}_i : i \notin \Omega\}$. Doing so would imply $Z+1$ firms with bids above $\max\{\overline{V}_i : i \notin \Omega\}$ so the price paid would be at least this value leading to a loss. On the other hand a high bidding firm is surely accepted at a price that is lower than $\min\{\underline{V}_i : i \in \Omega\}$ and hence always lower than its value. This price is unaltered by its actions (unless its bid is rejected), and thus the firm cannot improve. So we have established that high-low bidding will be a BNE. ■

Proof. (Lemma 14) If the bid of firm i is rejected with probability 1 for a range of signals $(x_i, \overline{V}_i)$ then there must be a set Ω of at least $\overline{Z}$ competitors that always bid $\overline{V}_i$ or higher. Otherwise there is some bid value $\overline{V}_i - \varepsilon$ that is accepted with positive probability and a range of signals close enough to $\overline{V}_i$ where the firm's value is above $\overline{V}_i - \varepsilon$ and so the firm would find it profitable to bid at

[40] It cannot be ruled out that the firm's bid is accepted, that the clearing price is set by a competitor's bid at the same price, and that such an event occurs with measure zero.

$\overline{V}_i - \varepsilon$, as these bids would be accepted with a positive probability and generate a positive expected payoff. A competitor firm $j \in \Omega$ bidding at or above $\overline{V}_i$ for all signals is only possible if the clearing price is $\underline{V}_j$, or lower, with probability 1, also for the lowest realization of Z, $\underline{Z}$. Otherwise there is a positive probability of the clearing price being at level $\underline{V}_j + \varepsilon$ or higher and the competitor would find it profitable to deviate and lower the bid for the signals in some range $(\underline{V}_j, \underline{V}_j + \varepsilon)$ to avoid that the bid is accepted at a clearing price above the value. Hence, at least $n - \underline{Z}$ firms in a set Ω_0 must, with probability 1, bid at $\min\{\underline{V}_j : j \in \Omega\}$, or lower. Moreover since Assumption B and our definitions imply that $\overline{V}_i > V_L$ and $V_L = \min\{\underline{V}_j : j \in \Omega_L\} \geq \min\{\underline{V}_j : j \in \Omega\}$, we deduce that the bids in Ω are strictly higher than any bids in Ω_0. Thus the two sets do not overlap giving a minimum of $n - \underline{Z} + \overline{Z}$ firms and hence $\overline{Z} = \underline{Z}$ and Ω_0 is the set complement of Ω. Since all the firms in Ω_0 have bids that are rejected with probability 1, we deduce that the bids for firms in set Ω must bid at $\max\{\overline{V}_i : i \in \Omega_0\}$ or higher. Thus we have established all the features of the high-low equilibrium.

If the bid of firm i is accepted with probability 1, for a range of signals $(\underline{V}_i, x_i)$ then there must be a set Ω_0 of at least $n - \underline{Z}$ competitors that always bid $\underline{V}_i$ or lower. Otherwise there is some value $\underline{V}_i + \varepsilon$ with a positive probability of being the clearing price, in which case firm i would find it profitable to deviate and lower its bid for the signals in the range $(\underline{V}_i, \underline{V}_i + \varepsilon)$. This will avoid outcomes where its bid is accepted at a clearing price above its value. An equilibrium where the firms in Ω_0 bid at or below $\underline{V}_i$ for almost all signals is only possible if deviations to bids equal to value for a range of high signals would be rejected with probability 1. Thus we require that there is a set Ω of at least $\overline{Z}$ firms that almost always bid at $\max\{\overline{V}_j : j \in \Omega_0\}$ or higher. Note that Assumption B and our definitions imply that $\underline{V}_i < V_U = \max\{\overline{V}_j : j \notin \Omega_U\} \leq \max\{\overline{V}_j : j \in \Omega_0\}$. Hence there cannot be an overlap between the sets Ω and Ω_0 and there is a minimum of $n - \underline{Z} + \overline{Z}$ firms and hence $\overline{Z} = \underline{Z}$ and Ω_0 is the set complement of Ω. Since all the firms in Ω have their bids accepted with probability 1, we deduce that the firms in Ω_0 must bid at $\min\{\underline{V}_j : j \in \Omega\}$ or lower. Thus we have again established all the features of the high-low equilibrium. ■

Before proving Theorem 1, we need three preliminary Lemmas and some additional notation. We write B_i for the set of bids for firm i. There is a distribution of bid values on B_i and because of our assumptions on the structure of bids this will consist of some intervals with a positive density and some points where there is a positive probability mass (occurring where the bid function is constant on an interval). From our assumption on the characteristics of the distribution of signals, the intervals with positive density and the points of positive probability mass on B_i remain the same no matter what signals are received by other bidders, or the value of Z.

In an equilibrium a bid y made by firm i is called active if it is accepted and rejected with positive probabilities in equilibrium. This property is determined by the characteristics of the probability distribution of the other firms' bids. Hence from our assumptions the activity status of a bid at some price y is independent of the firm's signal x. If there are more than one signal leading to a bid at y, then all of those bids will have the same activity status.

A bid at y by firm i is not active if there are guaranteed to be at least $\overline{Z}$ firms with bids strictly higher than y, since then the bid y is always rejected. We say firm j strictly dominates y if there is a probability of 1 of bids by firm j above y. So the condition for almost surely rejection of the bid y is that there are $\overline{Z}$ or more firms other than i, which strictly dominate y. Similarly a bid at y by firm i is not active if there are guaranteed to be at least $n - \underline{Z}$ firms with bids strictly lower than y, since then the bid y is always accepted. We say that y strictly dominates firm j if there is a probability of 1 of bids below y by firm j. So the condition for almost surely acceptance of the bid y is that y strictly dominates $n - \underline{Z}$ or more firms other than i.

Thus a bid of y by firm i is active if there are strictly less than $\overline{Z}$ firms j (with $j \neq i$) which strictly dominate y, and strictly less than $n - \underline{Z}$ firms j (with $j \neq i$) that y strictly dominates.

Lemma 21 *An active bid is price setting (the highest rejected bid) with a positive probability and the lowest bid that is accepted with a positive probability.*

Proof. Consider an active bid by firm i at a price y. Thus there are k_1 firms j (with $j \neq i$) which strictly dominate y, and k_2 firms j (with $j \neq i$) that y strictly dominates, and $k_1 < \overline{Z}$, $k_2 < n - \underline{Z}$. To show this bid is price setting we want a Z so that with a positive probability there are exactly Z other firms bidding above or at y. There are $n - 1 - k_2$ other firms that may bid at or above y, and k_1 that bid above y with probability 1. Thus we may choose any Z value between k_1 and $n - 1 - k_2$, and find a positive probability of exactly that number of bids at or above y. Note that $n - 1 - k_2$ is at least as large as k_1 since $n - 1 \geq k_1 + k_2$ as firms cannot be both strictly dominated by y and strictly dominate y. But $k_1 < \overline{Z}$, and $n - 1 - k_2 \geq \underline{Z}$ and so this range includes values in the range $\underline{Z}$ to $\overline{Z}$, and hence a value of Z that occurs with positive probability. It follows from our assumptions that this probability is positive independent of the signals that the bidders observe. Also we can show that the bid at y is the lowest bid that is accepted with a positive probability. For this we need exactly $Z - 1$ other firms bidding above or at y with a positive probability. For any Z value between $k_1 + 1$ and $n - k_2$ we will have exactly that number of bids at or above y with a positive probability. We have $k_1 + 1 \leq \overline{Z}$ and $n - k_2 > \underline{Z}$, so as before this range of values must include a value of Z that occurs with positive probability. ■

We are now ready to give the key technical Lemma that we use repeatedly in our analysis of equilibrium solutions. The result may perhaps seem obvious, given our ex-post optimality result in Lemma 13. But there is a subtlety that complicates the argument. Properties of a competitor's active bid could change once we condition on a firm making a bid at a specific price, such as y. In this case, a competitor's active bid below y may not be accepted with a positive probability and a competitor's active bid above y may not be rejected with a positive probability.

Lemma 22 *In an equilibrium with private values, if a firm's bid differs from the firm's valuation, there cannot be a positive probability of a bid from a competitor* *that is either equal to the firm's bid or strictly between the firm's bid and its valuation if either the firm's bid or the competitor's bid is active.*

Proof. Below we prove two properties of active bids. If we condition on a firm making a bid at y, then 1) a competitor's active bid below y is price setting with a positive probability and 2) a competitor's active bid above y is the lowest bid that is accepted with a positive probability. These two properties are sufficient for the case when the bid from the competitor is active. As explained below, the result for the case when the bid of the firm is active follows straightforwardly from Lemma 21.

We start by considering the case where firm i observes the value x and makes an equilibrium bid at $y > x$. Suppose that firm j has, with positive probability, an active bid between x and y, or at y. Thus we consider the event that some firm $j \neq i$ bids at a with $x < a \leq y$ and there are k_1 firms k (with $k \neq j$) which strictly dominate a, and k_2 firms k (with $k \neq j$) that a strictly dominates, where $k_1 < \overline{Z}$ and $k_2 < n - \underline{Z}$ (because the bid is active).

When $y > a$ and there are exactly Z firms bidding above a (including the bid by firm i), a will be the clearing price, firm i's bid is accepted and we get a contradiction from ex-post optimality. If firm i strictly dominates a then such a contradictory outcome is possible if $n - 1 - k_2 \geq Z \geq k_1$. Since $n - 1 - k_2 \geq \underline{Z}$ and $k_1 < \overline{Z}$, this range must intersect with a possible value of Z occurring with a positive probability. In the case that firm i does not strictly dominate a then we get a contradiction if there are exactly $Z - 1$ other firms, in addition to firm i (we condition on x), bidding above a. Excluding firm i there is a positive probability of a bid greater than a by exactly ℓ firms for any $\ell \geq k_1$ and $\ell \leq n - 2 - k_2$. We get a contradiction with a positive probability for $n - 2 - k_2 \geq Z - 1 \geq k_1$. Again these inequalities must be satisfied for a possible value of Z between $\underline{Z}$ and $\overline{Z}$. Thus in either case we have a contradiction.

When $y = a$, if we cannot use the argument above for $y > a$ then there is a positive probability of a competitor's bid being at $a = y$. We want to show that, conditioning on firm i's bid at y, there is a positive probability of firm i's bid being accepted with a clearing price at a, which would not be ex-post optimal. This will happen if amongst firms not i or j there are at least $Z - 1$ firms which bid at a or above (i.e. no more than $n - Z - 1$ bidding strictly below a) and no more than $Z - 1$ firms bidding strictly above a. Notice that firms i and j do not strictly dominate a and are not strictly dominated by a, so that we know $k_1 + k_2 \leq n - 2$. We require $k_2 \leq n - Z - 1$ and $k_1 \leq Z - 1$, i.e. $n - 1 - k_2 \geq Z \geq k_1 + 1$. Since $n - 1 - k_2 \geq \underline{Z}$ and $k_1 + 1 \leq \overline{Z}$, this range must intersect with a possible value of Z occurring with a positive probability, which gives the contradiction we require.

A similar argument applies when the firm bids below its value. Ex-post optimality shows that there cannot be a positive probability of the lowest accepted bid being between the bid and its value, and the argument proceeds similarly.

Now consider the case where the firm's bid itself is active. We start with the case where the firm bids above its value and the competitors bid is between the firm's bid and its value. From Lemma 21, there is a positive probability that the firm's bid is the lowest accepted bid, then (with a positive probability) the price is at the other firm's bid or above and is above the firm's value. Thus an improvement is possible. Similarly if the firm bids below its value and is active then there is a positive probability that the firm's bid is the highest rejected bid. This will lead to an improvement from raising the firm's bid to its value. ■

For the following Lemma and the proof of Theorem 1, it is helpful to define critical values a^- and a^+ which determine which bids are active in an equilibrium.We let $p_{[j]} = \inf\{B_j\}$, where B_j is the set of bids of firm j, and let a^- be the $\overline{Z}$th element $p_{[j]}$ if these are put in non-increasing order. Similarly we let $p^{[j]} = \sup\{B_j\}$ and let a^+ be the $n - \underline{Z}$ element $p^{[j]}$ if these are put in non-decreasing order.

Lemma 23 *If the inequality $a^- \leq a^+$ holds, then any bid y with $a^- < y < a^+$ is active. Moreover a bid at $a^- < a^+$ by firm i is active if there are less than $\overline{Z}$ firms j (with $j \neq i$), either with $p_{[j]} > a^-$, or with $p_{[j]} = a^-$ and not having an accumulation of bids at a^-. Also a bid at $a^+ > a^-$by firm i is active if there are less than $n - \underline{Z}$ firms j (with $j \neq i$), either with $p^{[j]} < a^+$, or with $p^{[j]} = a^+$ and not having an accumulation of bids at a^+. If $a^- = a^+$ then a bid by firm i at this common price is active if the conditions for both a^- and a^+ hold. A bid of y with $y < a^-$ is rejected with certainty, and is not active. A bid of y with $y > a^+$ is accepted with certainty, and is not active.*

Proof. Consider a bid at y by firm i. If $y < a^-$ then the bids from all the firms with $p_{[j]} \geq a^-$ are always higher than y. There are at least $\overline{Z}$ such firms and hence the bid y is certain to be rejected and is therefore not active. The situation with $y = a^-$ is more complicated since we may have more than one firm with $p_{[j]} = a^-$ (including firm i itself). A bid at a^- is rejected with probability 1 if there are at least $\overline{Z}$ firms other than i, either with $p_{[j]} > a^-$ or with $p_{[j]} = a^-$ and not having an accumulation of bids at a^-. When $y = a^-$ the bid is accepted with positive probability if there are less than $\overline{Z}$ firms other than i, either with $p_{[j]} > a^-$ or with $p_{[j]} = a^-$ and not having an accumulation of bids at a^-. If $y > a^-$ then the bid is accepted with positive probability.

If $y > a^+$ then the bids from all the firms with $p^{[j]} \leq a^+$ are always lower than y. There are at least $n - \underline{Z}$ such firms and hence the bid y is certain to be accepted, and is not active. A bid at a^+ is accepted with probability 1 if there are at least $n - \underline{Z}$ firms other than i, either with $p^{[j]} < a^+$ or with $p^{[j]} = a^+$ and not having an accumulation of bids at a^+. When $y = a^+$ the bid is rejected with positive probability if there are less than $n - \underline{Z}$ firms other than i, either with $p^{[j]} < a^+$ or with $p^{[j]} = a^+$ and not having an accumulation of bids at a^+. If $y < a^+$ then there is a positive probability that the bid is rejected. Together with the earlier observation on $y > a^-$ implying a positive probability of acceptance, we have shown a bid y with $a^- < y < a^+$ must be active. Combining all these implications gives the statement of the Lemma. ■

Proof. (**Theorem 1**) In the proof we will make use of a^-, a^+, $[j]$, $p_{[j]}$which are defined above Lemma 23. We start by establishing some inequalities: (i) $a^- < a^+$; (ii) $V_L < a^+$; and (iii) $V_U > a^-$.

(i) $a^- < a^+$. Since the equilibrium is not of high-low type, then from Lemma 14 for any firm the bid is accepted with positive probability for the highest signals, and rejected with positive probability for the lowest signals. If $a^- > a^+$ then since there are always $\overline{Z}$ bids above or at a^- the $n-\underline{Z}$ firms bidding at or below a^+ have bids rejected with certainty even for the highest signal, and we get a contradiction. Hence $a^- \leq a^+$.

Suppose that $a^- = a^+$, and write a^0 for this common price. Then since all firms have bids rejected with positive probability for the lowest signal, the $\overline{Z}$ firms bidding above or at a^0 must all have bids at a^0 with positive probability. Similarly, since the $n - \underline{Z}$ firms bidding below or at a^0 have bids accepted with positive probability for the highest signal, these firms must all bid at a^0 with positive probability. Thus every firm has positive probability of bidding at a^0 and all these bids are active. To avoid a contradiction from Lemma 22 firms with values strictly above a^0 must bid strictly above a^0 and firms with values strictly below a^0 must bid strictly below a^0. Thus the $\overline{Z}$ firms bidding at a^0 or above have values at a^0 or above, and the $n - \underline{Z}$ firms bidding at or below a^0 have values at a^0 or below. But this contradicts $V_L < V_U$ which is an implication of Assumption B. Thus we have $a^- < a^+$.

(ii) $V_L < a^+$. Suppose otherwise and $V_L \geq a^+$. Then there are $\overline{Z}$ firms, for which all values are at a^+ or above. Since $\overline{Z} \geq 2$ (Assumption A), there are at least two such firms. Since each of these two have bids rejected with positive probability, then the two firms have ranges of values, (say $[v_x, v_y]$ for the first with $a^+ < v_x < v_y$, and $[u_x, u_y]$ for the second with $a^+ < u_x < u_y$) for which these firms bid at a^+ or below (according to Lemma 23) and are rejected with a positive probability. But if any bid of the first firm has a positive probability of being accepted for signals in $[v_x, v_y]$ then this bid is active and we obtain a contradiction from Lemma 22 which can be applied both when the second firm has bids from $[u_x, u_y]$ at or below the active bid of the first firm and also when these bids are in between the active bid of the first firm and a^+.[41] Thus for signals in $[v_x, v_y]$ the first firm bids low enough (at or below a^-) to have zero probability of acceptance. It follows from the definition of a^- that there are always $\overline{Z}$ bids at or above a^-, and that at most $\overline{Z} - 1$ bidders always bid above $a^- + \varepsilon$ for any $\varepsilon > 0$. This implies that, with a positive probability, the clearing price is at a^- or below. We have $a^- < a^+ < v_x < v_y$, so this gives a contradiction to Lemma 13 and ex-post optimality for the first firm for values in the range $[v_x, v_y]$.

(iii) $V_U > a^-$. This is established using a similar argument. If this does not hold and $V_U \leq a^-$ then there are at least $n - \underline{Z}$ firms with all values less than or equal to a^-. Assumption A implies that there are at least two such firms. Since each of these two have bids accepted with a positive probability for their high bids, then the two firms have ranges of values (say $[v_x, v_y]$ for the first with $v_x < v_y < a^-$, and $[u_x, u_y]$ for the second with $u_x < u_y < a^-$) for which these firms bid at a^- or above (according to Lemma 23) and are accepted with positive probability. But if any bid of the first firm has a positive probability of being rejected for signals in $[v_x, v_y]$ then this bid is active and we have a contradiction from Lemma 22 which can be applied both when the second firm has bids from $[u_x, u_y]$ at or below the

[41] Recall that, by assumption, all bids from the second firm are at or below a^+ for values in the range $[u_x, u_y]$.

active bid of the first firm and also when these bids are above the active bid of the first firm. Thus for signals $[v_x, v_y]$ the first firm bids high enough (at or above a^+) to have zero probability of being rejected. It follows from the definition of a^+ that there are $n - \underline{Z}$ bidders that always bid at or below a^+, and that at most $n - \underline{Z} - 1$ bidders always bid below $a^+ - \varepsilon$ for any $\varepsilon > 0$. This implies that, with a positive probability, the clearing price is less than or equal to the first firm's bid when its value is in the range $[v_x, v_y]$. But this gives a contradiction to Lemma 13 and ex-post optimality.

We begin by proving part (a) which we accomplish in a number of steps. Define $v_0 = \max(a^-, V_L)$ and $v_1 = \min(a^+, V_U)$. We have established that $a^- < V_U$ and it follows from Assumption B that $V_L < V_U$, so we have $v_0 < V_U$. But we also have $V_L < a^+$ and $a^- < a^+$, so $v_0 < a^+$ and we deduce $v_0 < v_1$.

Step 1. We show there are bids in the intervals just above a^- and just below a^+. We will prove this for a^-. Recall that by definition (see just above 23), a^- is the infimum of bids made by firm $[\overline{Z}]$. If this firm has no accumulation of bids at a^-, then it follows from the construction of regular bids that firm $[\overline{Z}]$ must have bids in the intervals just above a^-, and we are done. The same argument can be made for any other firm that has a^- as the lower bound on its bids. Hence, if there are no bids in the interval just above a^-, then firm $[\overline{Z}]$, and any other firm that has a^- as its lower bound on bids, must have an accumulation of bids at a^-. From the ordering of the firms (see just above 23), it follows that at most $\overline{Z} - 1$ firms bid strictly above a^- with certainty, so the accumulated bids at a^- are accepted with a positive probability. Moreover, we have assumed that all firms have bids accepted with positive probability for the highest signals. Hence, all firms bid at or above a^- with a positive probability. Thus bids at a^- are active (see definition just above Lemma 21) since they are rejected and accepted with positive probabilities. A similar argument can be made for a^+. If there is an accumulation of bids at a^+, then those bids must be active.

Let g_1 be the infimum of the bids in (a^-, a^+) and take $g_1 = a^+$ if there are no such bids. We want to show that $g_1 = a^-$. Suppose this fails and $g_1 > a^-$, so there is an interval (a^-, g_1) without bids. If $g_1 < a^+$ then it follows from Lemma 23 that there are active bids at g_1 or just above. The definition of a^+ implies that there is an accumulation of bids at a^+ or bids just below a^+. Hence, if $g_1 = a^+$ then there must be an accumulation of bids at a^+, which (according to our result above) are active. From assumption A and the definition of a^- there are (at least) two firms j_1 and j_2 (indexed by $\overline{Z} + 1$, and $\overline{Z} + 2$ when $p_{[j]}$ are put in non-increasing order) with $\inf\{B_{j_1}\} \leq a^-$ and $\inf\{B_{j_2}\} \leq a^-$. Because their high bids are accepted with positive probability, these two firms both have bids greater than or equal to a^- with positive probability. Thus they each have a range of values for which they bid at a^- or above. If for either firm this range of values includes values strictly less than a^- then since there are active bids at a^- from firm $[\overline{Z}]$, we get a contradiction from Lemma 22. Thus the range of values for each firm j_1 and j_2 has a supremum strictly greater than a^-.Consider the bids for these two firms for values in the range(a^-, g_1). Since there are active bids at a^- from firm $[\overline{Z}]$, we get a contradiction from Lemma 22 if either of these firms bid at a^- or below in this range of values. Also if either firm bids (strictly) higher than g_1, say at $g_1 + \delta$, in this range of values then there is positive probability of active bids in the range $[g_1, g_1 + \delta)$. Again this gives a contradiction from Lemma 22. Thus we deduce that both firms must bid at $g_1 < a^+$ with positive probability in this range of values. These are active bids and this contradicts Lemma 22. Hence we have established that $g_1 = a^-$.

Step 2. We show that there are bids throughout (v_0, v_1).

We suppose there is a gap (g_3, g_4) with no bid in this interval, with $a^- \leq g_3 < g_4 \leq a^+$. We choose g_3 as small as possible and g_4 as large as possible subject to there being no bid in the interval (g_3, g_4) (so there are bids at g_3 or just below, and at g_4 or just above). The result of step 1 implies that this definition has $a^- < g_3$ and $g_4 < a_1^+$. Suppose that there is an intersection between (g_3, g_4) and the range (v_0, v_1).Then from Assumption C there are at least three firms with values in a subinterval X_A of (g_3, g_4). These firms cannot bid in (g_3, g_4) for values in X_A, so either at least two of them bid at g_3 or below, or at least two of them bid at g_4 or above for values in X_A. Suppose two bid at g_3 or below. We know that there is a bid at g_3 or just below, which may be a bid from one of these firms. Moreover this bid is active since $a^- < g_3 < a^+$. This gives a contradiction from Lemma 22 both in the case of bids at g_3 and also when the bids are strictly less than g_3. In the same way if two firms bid at g_4 or above for values in X_A, we get a contradiction since there is an active bid at g_4 or just above (possibly from one of these two firms). Thus we have shown that any gap (g_3, g_4) is outside the interval (v_0, v_1). We have by definition that $(v_0, v_1) \subseteq (a^-, a^+)$, so there are no gaps in (v_0, v_1) that are outside (a^-, a^+). This establishes this step of the proof.

Step 3. We show that each firm bids at value for values in the range (v_0, v_1) and bids at or below v_0 for values below v_0 and at or above v_1 for values above v_1.

The bids throughout (v_0, v_1) are all active. If a firm with value at say $y \in (v_0, v_1)$ makes a bid that is not equal to y there will be active bids either just below y or just above y between the firm's bid and its value. This gives a contradiction to Lemma 22. For similar reasons, a firm cannot bid above v_0 for values below v_0 and a firm cannot bid below v_1 for values above v_1.

Step 4. We show that $V_L \leq a^-$ so that $v_0 = a^-$

Suppose that $V_L > a^-$ and we will obtain a contradiction. Let p_L be the infimum of bids made by firms in the set Ω_L, which has size $\overline{Z}$, and let $i_L \in \Omega_L$ have $\inf\{B_{i_L}\} = p_L$. We have $\overline{Z} - 1$ other firms j in Ω_L with $\inf\{B_j\} \geq p_L$ and so from the definition of a^- we can deduce $p_L \leq a^-$. It follows from the definition of V_L that the firm i_L has values greater than or equal to V_L and so from our assumption on the regularity of bids, there is a range of values (a_L, b_L), where $a_L > V_L$, for which bids are strictly below value. $V_L > a^-$ implies that $v_0 = V_L$, so we know that there are active bids just above V_L (from Step 3). According to Assumption C this includes bids from a competitor to i_L, so we have a contradiction from Lemma 22.

Step 5. We show that $a^- \leq V_L'$

We suppose that $a^- > V_L'$ and obtain a contradiction. The definition of V_L' means that any set of $\overline{Z}$ firms contains at least two with $\underline{V}_j \leq V_L'$. We consider the $\overline{Z}$ firms with the highest values of $p_{[j]}$ (see definition just before Lemma 23). Amongst this set of firms, which by definition have $p_{[j]} \geq a^-$, there are at least two with $\underline{V}_j \leq V'_L < a^-$. Both these firms bid above their values for a range of signals below a^-. If either of them bid strictly above a^- for signals in this range then there is a contradiction from Lemma 22, since we know (Step 1) that there are active bids just above a^-. Thus both have an accumulation of bids at a^- which are active from our observation earlier. This gives a contradiction from Lemma 22. Note that this upper bound on a^- is stronger that the bound we proved above in (iii) that $a^- < V_U$ since, from Assumption B, $V'_L < V_U$.

Step 6 We complete the proof of part (a)

At this point we have shown that $V_L \leq a^- \leq V'_L$. We can use this to establish that $V'_U \leq a^+ \leq V_U$. We consider the reflected equilibrium. The value of a^- in the reflected equilibrium is $-a^+$, and the values of V_L and V'_L in the reflected equilibrium are $-V_U$ and $-V'_U$ respectively. Hence we can apply what we have shown to the reflected equilibrium to establish that $-V_U \leq -a^+ \leq -V'_U$. Thus $v_1 = a^+$, and we have established that it is necessary for an equilibrium that is not of the high-low type to satisfy the properties described in part (a).

Now we turn to part (b). When $v_0 = V_L$ then $a^- = V_L$. It follows from Step 3 that all firms bid at v_0 or lower for signals strictly less than v_0.

For part (c) We suppose $v_0 > V_L$ and hence $v_0 = a^- > V_L$. We consider the $\overline{Z}$ firms with the highest values of $p_{[j]}$, and hence with all bids at or above a^-. From the definition of V_L we see that there is at least one firm i_X in this set with $\underline{V}_{i_X} \leq V_L$, which bids above value for values in the range $(\underline{V}_{i_X}, a^-)$. It follows from Step 3 that firm i_X must bid at a^- for values in the range $(\underline{V}_{i_X}, a^-)$. The condition $a^- > V_L$ implies that at most $\overline{Z} - 1$ firms have all values above a^-. It follows from Step 3 that other firms bid at a^- or lower with a positive probability. This means that bids at a^- are active. But to avoid a contradiction from Lemma 22 no competitor of firm i_X can bid in the range $\left(\underline{V}_{i_X}, a^-\right]$ with a positive probability. Hence, except for firm i_X, all firms must bid at $\underline{V}_{i_X}$ or lower for values below a^-.

It is straightforward to show that parts (d) and (e) follow from parts (b) and (c) applied to the reflected auction (in the same way that we did for Step 6 of part (a)).

Finally we show that a set of bids satisfying these conditions must be an equilibrium, since bids are all ex-post optimal. Suppose that a firm has a signal between v_0 and v_1 and bids its value, then these bids are always ex-post optimal, and thus we only need to consider bids not at value outside this range. Consider the case that $v_0 = V_L$ and suppose that a firm observes a signal less than v_0 then, since all bids between v_0 and v_1 are made at value, with probability 1 there are $\overline{Z}$ bids strictly above v_0. Thus every bid at or below v_0 is rejected with certainty, and so bidding at $y \leq v_0$ for a signal less than v_0 is ex-post optimal. Suppose that $V_L < v_0 \leq V'_L$ and we consider firm i_X bidding at v_0 for values below v_0. This is the only firm with an accumulation of bids at this price. If the bid is rejected, the price will be at v_0 or above. If the bid is accepted then the clearing price is with probability 1 set by one of the firms with values below v_0 all of whom bid below

$\underline{V}_{i_X}$, so that the clearing price cannot be higher than the value for i_X and we have ex-post optimality for firm i_X. Now consider firms bidding below $\underline{V}_{i_X}$ for the case $V_L < v_0 \leq V_L'$. Since there are $\overline{Z} - 1$ firms who bid above V_L' with probability 1 and firm i_x bids at v_0 or above, firms bidding below $\underline{V}_{i_X}$ have their bids rejected. Since the last accepted bid is at v_0 or above, which is higher than their value, then these bids are also ex-post optimal. We can use the reflected equilibrium to show that there is also no improvement possible at the top of the bidding range, where v_1 takes the place of v_0. ■