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Interconnected Conflict*

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Abstract

We study a model of conflict with multiple battlefields and the possibility of investments spillovers between the battlefields. Results of conflicts at the individual battlefields are determined by the Tullock contest success function based on efforts assigned to a battlefield as well as efforts spilling over from the neighbouring battlefields. We characterize Nash equilibria of this model and uncover a network invariance result: equilibrium payoffs, equilibrium total expenditure, and equilibrium probabilities of winning individual battlefields are independent of the network of spillovers. We show that the network invariance holds for any contest success function that is homogeneous of degree zero and has the no-tie property. We define a network index that characterizes equilibrium efforts assignments of the players. We show that the index satisfies neighbourhood inclusion and can, therefore, be considered a network centrality.

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1 Introduction

Models of conflicts with multiple battlefields are one of the oldest studied in modern game theory, starting with the Colonel Blotto game of Borel (1921). Applications of this model range widely and include military conflicts (Garfinkel and Skaperdas, 2012; Kovenock and Roberson, 2018), political competition (Brams and Davis, 1974; Snyder, 1989), advertising competition (Friedman, 1958), and network security (Dziubiński et al., 2016). As a result, the study of inter-connected contests remains a very active field of study.

An aspect of these conflicts, usually omitted in the literature, is the possibility of spillovers of efforts assignments between the battlefields. For instance, in advertising competition there is a phenomenon of cross media advertising spillover: investments in advertising in one media channel can impact or influence other media channels (Naik and Raman, 2003; Sridhar et al., 2022). Similarly, in political campaigning expenditures in one constituency may have a bearing on neighbouring constituencies. In military applications, allocation of forces in one location may have impact on neighbouring battlefields. The aim of this paper is to study competition and conflict in such settings.

We consider a problem of conflict with multiple battlefields with the possibility of spillovers of investments between the battlefields. There are two parties facing a number of battles. Each battle has a value, common to the two parties, and an investment on a battle may have an indirect effect on other battles. The degree of this effect varies across the battles. The parties take these investment externalities into account when choosing their investment levels. The probability of winning the conflict at a given battlefield is determined by a contest success function (CSF) and depends on the amount of direct investment to the battlefield *plus* the amount of indirect investment spilling over from other battlefields. We study Nash equilibrium of the conflict game and in particular how the network of spillovers affects the allocation of resources and the outcomes for the two parties.

We start with a preliminary observation that Nash equilibrium exists and is (generically) unique. Let us define the effective investment in a battle as the sum of direct investment in that battle plus the investment in neighbouring battles. We show that in equilibrium each battlefield receives a positive effective investment from both players. However, in general, only a subset of battlefields receive positive direct investment. The set of battlefields receiving zero investment is the same for each of the two players.

We then turn to the effects of networks on allocations and earnings and obtain a *network invariance result*: equilibrium payoffs, equilibrium total expenditure, and equilibrium probabilities of winning individual battlefields are independent of the network of spillovers. This means that as we move across networks, the players adjust their equilibrium investment strategies to keep these quantities unchanged.

In equilibrium, individual efforts vary across nodes in a network. Our final result provides a characterization of allocations across battles using a network index. The vector of equilibrium investment levels is obtained by scaling this index by a factor that depends on the costs of investments and the CSF. The network index is a solution to a non-linear optimization problem defined only by the network of spillovers. We show that the index is a generalization – to directed, weighted networks – of *neighbourhood inclusion* (Schoch and Brandes, 2016). This is considered a minimal property for a network index to be a network centrality.

1.1 Related literature

Our work is related to the literature on games on multiple battlefields that started with Borel (1921). There is a vast body of work that spans disciplines like economics, computer science, international relations, and operations research; for surveys see Kovenock and Roberson (2012), Garfinkel and Skaperdas (2012), Dziubiński et al. (2016), and Goyal (2023).

Let us start by discussing the closely related models with two players facing a common set of battlefields. Our paper generalizes the model of Friedman (1958). In his model, the winner of a battlefield is determined by lottery dependent on individual efforts of the players on that battlefield. Adopting Tullock's contest success function, Friedman (1958) obtained characterization of Nash equilibrium in the case of all the battlefields having equal value. More recent work has studied the impact of richer formulations of the payoffs as a function of battles won. Our work extends this model by allowing for heterogeneous values across battlefields and spillovers between efforts at individual battlefields.

In the context of political competition, Brams and Davis (1974) and Snyder (1989) consider the effect of different objectives on equilibrium play: winning the most districts versus winning a majority of districts. In more recent research, Kovenock and Roberson (2018) similarly study the effects of changing objectives on equilibrium of the game: they consider a model in which individual payoffs depend on weakest link (the defender earns positive payoffs only if she wins all nodes) and best-shot payoffs (the winner earns positive payoff as long as she wins one battle). Our contribution to this literature is that we generalize the effects of allocations and allow for spillovers across individual districts. We characterize equilibrium in these networked conflict model and obtain a network invariance result. To the best of our knowledge these results are novel.

There is also a large literature on conflict with a quite different set of models in which players are nodes of a network and each player is engaged in battle(s) with players to whom they are linked, see, for instance, Franke and Öztürk (2015). Xu et al. (2022) and Matros and Rietzke (2022) study a generalization of Franke and Öztürk (2015) where each player can engage in a set of battles and each battle can involve two or more players. The main difference between these papers and our paper is existence of explicit effort spillovers in our case. We build on the methods in Xu et al. (2022) to obtain existence and uniqueness of Nash equilibrium. The novelty of our work lies in extending the uniqueness result to corner equilibria, where some battlefields receive zero efforts from the players, the characterization of the equilibrium in terms of a network index, and in the network invariance result.

We would also like to mention two other recent papers, Boosey and Brown (2022) and König et al. (2017). In particular, König et al. (2017) consider a model in which nodes choose efforts and the links with other nodes are signed and indicate positive or negative spillovers of effective efforts. The methods of analysis and the main results concerning characterization of equilibrium efforts are different to ours. In their model effort is proportional to Bonacich centrality, we obtain a network index that is quite different. In their model networks make a big difference to equilibrium payoffs of different players, while we show that payoffs are network invariant. They also restrict attention to interior equilibria only while our analysis covers interior as well as corner equilibria.

The rest of the paper is organized as follows. In Section 2 we define the model. In Section 3 we present the results. In particular, in Section 3.3 we study the sufficient conditions for the network invariance results and in Section 3.4 we study the network index related to the equilibrium efforts. All the proofs are given in the Appendix.

2 Model

Two players, 1 and 2, compete on a set $B = \{1, \ldots, m\}$ of m battlefields, each battlefield $k \in B$ of value $v^k > 0$, common to both players. Each player $i \in \{1, 2\}$ chooses a vector $\boldsymbol{e}_i = (e_i^k)_{k\in B} \in \mathbb{R}^B_{\geq 0}$ of non-negative efforts across the battlefields. Effort is costly and each player $i \in \{1, 2\}$ faces a constant marginal cost of effort, $c_i > 0$, that can be different across the players. There is a network of non-negative effort spillovers between the battlefields, represented by the adjacency matrix $\boldsymbol{\rho} = (\rho_{k,l})_{k,l\in B}$, $\rho_{k,l} \geq 0$ for all $(k, l) \in B^2$. Assignment of effort e^k to battlefield k results in spillover $\rho_{k,l}e^k$ to battlefield $l \in B$. The vector of effort assignments to all the battlefields, $\boldsymbol{e} = (e^k)_{k\in B}$ results in a vector of *effective efforts* due to network spillovers, $\boldsymbol{y} = (y^k)_{k\in B}$, with $y^k = e^k + \sum_{l\in B\setminus\{k\}} \rho_{l,k} e^l$. In matrix notation

$$\boldsymbol{y} = \left(\mathbf{I} + \boldsymbol{\rho}^T\right)\boldsymbol{e},$$

where $\mathbf{I}$ denotes the identity matrix.

The probabilities of players winning the contest at battlefield $k \in B$, given the effective efforts assignments to k, (y_1^k, y_2^k), are determined by a contest success function (CSF), $p : \mathbb{R}^2_{\geq 0} \to [0,1]^2$. The probability of player $i \in \{1,2\}$ winning the contest at battlefield $k \in B$ is $p_i(y_1^k, y_2^k)$. Throughout most of the paper we focus on the Tullock contest success functions which have the form

$$p_i(y_1, y_2) = \frac{(y_i)^\gamma}{(y_1)^\gamma + (y_2)^\gamma},$$

with $\gamma \in (0,1]$. The assumption that $\gamma \leq 1$ is fairly standard in the contest literature: it is made to ensure the existence of pure strategy Nash equilibria. When $\gamma > 1$, payoffs cease to be concave and, depending on the costs and values of battlefields, equilibria in pure strategies may not exist; for a discussion of some of the issues that arise for large γ, see Baye et al. (1994) and Ewerhart (2015).

Given the pair of efforts $(\boldsymbol{e}_1, \boldsymbol{e}_2)$, the expected payoff to player $i \in \{1,2\}$ is

$$\Pi_i(\boldsymbol{e}_1, \boldsymbol{e}_2) = \sum_{k\in B} v^k p_i^k(\boldsymbol{e}_1, \boldsymbol{e}_2) - c_i \sum_{k\in B} e_i^k, \tag{1}$$

where

$$p_i^k(\boldsymbol{e}_1, \boldsymbol{e}_2) = p_i(y_1^k, y_2^k) \tag{2}$$

is the probability of player i winning battlefield k, given the efforts profile $(\boldsymbol{e}_1, \boldsymbol{e}_2)$.

A strategy profile $(\boldsymbol{e}_1^*, \boldsymbol{e}_2^*) \in \mathbb{R}^B_{\geq 0} \times \mathbb{R}^B_{\geq 0}$ is a pure strategy Nash equilibrium of this interconnected conflict game if, for any player $i \in \{1,2\}$ and any $(\boldsymbol{e}_1, \boldsymbol{e}_2) \in \mathbb{R}^B_{\geq 0} \times \mathbb{R}^B_{\geq 0}$,

$$\Pi_i(\boldsymbol{e}_1^*, \boldsymbol{e}_2^*) \geq \Pi_i(\boldsymbol{e}_i, \boldsymbol{e}_{-i}^*).$$

We are interested in the properties of pure strategy Nash equilibria.

3 Analysis

We start the analysis by providing characterization of Nash equilibria in our model. In the analysis we focus on the generic case when the matrix $\mathbf{I} + \boldsymbol{\rho}$ is non-singular. As it turns out, this guarantees uniqueness of Nash equilibrium.

Before we turn to the general model, we provide two examples illustrating Nash equilibria of the game with simple networks of spillovers.

3.1 Examples

Example 1 (Regular network)**.** *Consider a directed regular network $\boldsymbol{\rho}$, i.e. a network where the total weight on incoming links is the same for all the nodes and the total weight of outgoing links is the same for all the nodes (c.f. Figure 1). Formally, the adjacency matrix of the network satisfies $\boldsymbol{\rho}\mathbf{1} = \boldsymbol{\rho}^T\mathbf{1} = d\mathbf{1}$, for some $d \geq 0$. Let*

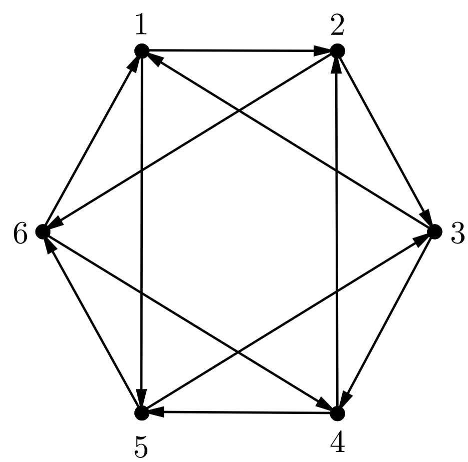

Figure 1: A regular network over 6 nodes with weight of each link equal to $d/2$.

$\gamma = 1$ and let all battlefields have the same value 1, $\boldsymbol{v} = (v^k)_{k\in B} = \mathbf{1}$. Let $\boldsymbol{x} = \mathbf{1}$.

If p is the Tullock CSF with parameter $\gamma = 1$ then strategy profile $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ with

$$\boldsymbol{e}_1 = \frac{c_2}{(c_1 + c_2)^2}\boldsymbol{x}, \qquad \boldsymbol{e}_2 = \frac{c_1}{(c_1 + c_2)^2}\boldsymbol{x}$$

is the unique Nash equilibrium of the game played on the regular network with homogeneous battlefield values. In the equilibrium all battlefields receive equal positive efforts from each players. The efforts are different across the players, unless they face the same costs of effort.

Aggregate equilibrium effort of player i is

$$\sum_{k\in B} e_i^k = n\frac{c_{-i}}{(c_1 + c_2)^2}$$

and the equilibrium payoff of players i is equal to

$$\Pi_i(\boldsymbol{e}_1, \boldsymbol{e}_2) = n\left(\frac{c_{-i}}{c_1 + c_2}\right)^2.$$

The probability of winning battlefield $k \in B$ by player i is

$$p_i^k(\boldsymbol{e}_1, \boldsymbol{e}_2) = \frac{c_{-i}}{c_1 + c_2}.$$

Example 2 (Star network)**.** *Consider a directed star network over n nodes, $\{0, \dots, n-1\}$, with centre 0 and links outgoing from the centre to the spokes, each edge from node 0 node $i \in \{1, \dots, n-1\}$ weighted with $\lambda_i > 0$, where $\lambda_1 \leq \dots \leq \lambda_{n-1}$ (c.f. Figure 2). The adjacency matrix of the network is*

$$\boldsymbol{\rho} = \left[\begin{array}{c|c} 0 & \boldsymbol{l}^T \\ \hline \boldsymbol{0} & \boldsymbol{0} \end{array}\right],$$

where $\boldsymbol{l} = [\lambda_1, \dots, \lambda_{n-1}]^T$. Value of each battlefield is equal to 1.

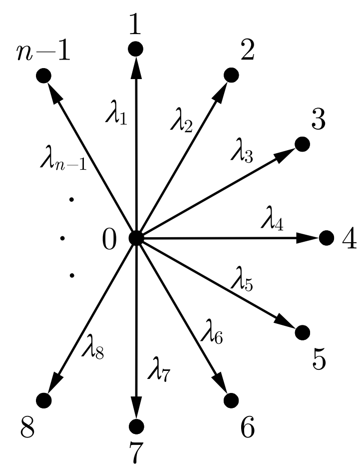

Figure 2: A star network with heterogeneous spillovers.

Let

$$m = \max\left(\left\{k \in \{1, \dots, n-1\} : \lambda_k < \frac{1 - \sum_{i=1}^{k-1} \lambda_i}{n-k+1}\right\} \cup \{0\}\right)$$

and let $\boldsymbol{x} = (x_0, \dots, x_{n-1})^T$ be such that, for $i \in \{0, \dots, n-1\}$,

$$x_i = \begin{cases} \frac{n-m}{1-\sum_{j=1}^m \lambda_j}, & \textit{if } i = 0, \\ 1 - \left(\frac{n-m}{1-\sum_{j=1}^m \lambda_j}\right)\lambda_i, & \textit{if } 1 \leq i \leq m, \\ 0, & \textit{otherwise.} \end{cases}$$

Notice that, by the definition of m,

$$1 - \sum_{i=1}^{m} \lambda_i > (n-m)\lambda_m > 0.$$

Hence $x_i > 0$, for all $i \in \{0, \ldots, m\}$. Moreover, $x_0 \geq x_2 \geq \cdots \geq x_{n-1}$ and $\sum_{i=0}^{n-1} x_i = n$.

If p is the Tullock CSF with parameter $\gamma = 1$ then strategy profile $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ with

$$\boldsymbol{e}_1 = \frac{c_2}{(c_1+c_2)^2}\boldsymbol{x}, \qquad \boldsymbol{e}_2 = \frac{c_1}{(c_1+c_2)^2}\boldsymbol{x}.$$

is the unique Nash equilibrium of the game played on the star network. In equilibrium the centre and m periphery nodes with the lowest spillover from the centre receive positive efforts from each of the players and the remaining periphery nodes receive zero effort from each of the players.

Aggregate equilibrium effort of player i is

$$\sum_{k \in B} e_i^k = n \frac{c_{-i}}{(c_1+c_2)^2}$$

and the equilibrium payoff of players i is equal to

$$\Pi_i(\boldsymbol{e}_1, \boldsymbol{e}_2) = n \left(\frac{c_{-i}}{c_1+c_2} \right)^2.$$

The probability of winning battlefield $k \in B$ by player i is

$$p_i^k(\boldsymbol{e}_1, \boldsymbol{e}_2) = \frac{c_{-i}}{c_1+c_2}.$$

These two examples bring out a number of points: one, we see that in the regular graph the allocations by the two player depends solely on the relative costs; two, in the star network we see that nodes with greater spillovers attract more resources from both players; three, a striking feature of equilibria in the two examples is that the aggregate equilibrium efforts, equilibrium payoffs and equilibrium winning probabilities for individual battlefields associated with a given player are the same for the two networks (if the numbers of battlefields are equal).

We now establish existence and uniqueness of equilibrium and develop a characterization of equilibrium that brings out the generality of these three observations across the set of all networks.

3.2 Equilibrium characterization

We show that Nash equilibrium exists and, as long as the matrix $\mathbf{I}+\boldsymbol{\rho}$ is non-singular, it is unique. Moreover the equilibrium has the following structure. There exists a set of battlefields, $P \subseteq B$, at which both players exert positive effort and both players exert 0 effort at the remaining battlefields. In addition, in equilibrium every battlefield receives positive effective effort from each player. Hence every battlefield is contested in equilibrium, however there may be battlefields that are contested only indirectly, through spillovers from other battlefields.

Before stating the result we introduce some notation. Given a vector $\boldsymbol{z} \in \mathbb{R}^B$ and a set $S \subseteq B$, we will use $\boldsymbol{z}^S = (z^k)_{k\in S}$ to denote the vector $\boldsymbol{z}$ restricted to the entries in S. Similarly, given a matrix $\boldsymbol{a} \in \mathbb{R}^{B\times B}$ and two sets $S \subseteq B$ and $T \subseteq B$, we will use $\boldsymbol{a}_{S,T} = (a_{k,l})_{k\in S, l\in T}$ to denote matrix $\boldsymbol{a}$ restricted to the entries in $S \times T$. In particular, matrix $\boldsymbol{\rho}_{S,T}$ is the adjacency matrix of the spillovers from the battlefields in S to the battlefields in T. Given a set $S \subseteq B$, we will also use $-S = B \setminus S$ to denote the complement of S. Lastly, recall, that given two vectors $\boldsymbol{z} \in \mathbb{R}^S$ and $\boldsymbol{y} \in \mathbb{R}^S$, $\boldsymbol{z} \oslash \boldsymbol{y} = (z_i/y_i)_{i\in S}$ is the Hadamard division of $\boldsymbol{z}$ by $\boldsymbol{y}$.

Theorem 1. *Let p be the Tullock CSF with $\gamma \in (0,1]$. Pure strategy Nash equilibrium exists. Moreover, if $\mathbf{I}+\boldsymbol{\rho}$ is non-singular then the equilibrium is unique and in equilibrium every battlefield receives positive effective effort from every player $i \in \{1,2\}$. A strategy profile $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ is the Nash equilibrium if and only if there exists a set of battlefields $P \subseteq B$ such that, for $i \in \{1,2\}$,*

$$\begin{aligned} \boldsymbol{e}_1^{-P} &= \boldsymbol{e}_2^{-P} = \mathbf{0}, \\ \boldsymbol{e}_i^{P} &= \frac{\gamma\,(c_1c_2)^\gamma}{c_i\,(c_1^\gamma + c_2^\gamma)^2}\left(\mathbf{I}+\boldsymbol{\rho}_{P,P}^T\right)^{-1}(\boldsymbol{v}_P \oslash \boldsymbol{\mu}_P), \end{aligned} \tag{3}$$

where $\boldsymbol{\mu}$ satisfies

$$\begin{aligned} (\mathbf{I}+\boldsymbol{\rho}_{P,P})\boldsymbol{\mu}_P + \boldsymbol{\rho}_{P,-P}\boldsymbol{\mu}_{-P} &= \mathbf{1} \\ \frac{1}{4}\boldsymbol{\rho}_{P,-P}^T\left(\mathbf{I}+\boldsymbol{\rho}_{P,P}^T\right)^{-1}(\boldsymbol{v}_P \oslash \boldsymbol{\mu}_P) &= \boldsymbol{v}_{-P} \oslash \boldsymbol{\mu}_{-P}. \end{aligned} \tag{4}$$

Moreover,

$$\begin{aligned} p_i^k(\boldsymbol{e}_1, \boldsymbol{e}_2) &= \frac{c_{-i}^\gamma}{c_1^\gamma + c_2^\gamma}, \text{ for all } k \in B, \\ \sum_{k\in B} e_i^k &= \frac{\gamma\,(c_1c_2)^\gamma}{c_i\,(c_1^\gamma + c_2^\gamma)^2}\mathbf{1}^T\boldsymbol{v}, \\ \Pi_i(\boldsymbol{e}_1, \boldsymbol{e}_2) &= \frac{c_{-i}^\gamma\,(c_1^\gamma + c_2^\gamma - \gamma c_i^\gamma)}{(c_1^\gamma + c_2^\gamma)^2}\mathbf{1}^T\boldsymbol{v}. \end{aligned}$$

The theorem states existence and uniqueness of Nash equilibrium, provides characterization of equilibrium efforts of the players, as well as characterization of equilibrium payoffs, equilibrium total expenditure of the players and equilibrium winning probabilities across the battlefields.

The standard method to obtain equilibrium efforts in contest models is solving the constrained optimization problem associated with the best response conditions. If the solution is interior, existence and uniqueness follows. In the model with spillovers, studied in this paper, corner solutions arise naturally and the main challenge in proving the theorem is to address such solutions. To address existence, we carefully adapt the proof technique for existence developed in Xu et al. (2022). Uniqueness is more challenging due to the possibility of corner solutions. Notice in particular, that a priori it is not even clear that in a corner equilibrium both players will assign zero effort to the same set of battlefields. We address this issue by constructing a symmetric strategy profile that is an equilibrium in the symmetric case where both players have costs equal to 1 and then explicitly constructing equilibria for the remaining values of costs by transforming the equilibrium obtained for the symmetric case. This construction shows that that there exist equilibria where the set of battlefields receiving zero effort is common for the two players. Using that fact and general properties of equilibria in contest models (Lemma 1 in the Appendix) we obtain uniqueness of equilibria in our model.

Using uniqueness and the fact that in equilibrium both players exert strictly positive efforts at the common set of battlefields, P, and they exert zero effort at the remaining battlefields, we obtain the characterization of equilibrium efforts. The equilibrium efforts are related to a vector $\boldsymbol{\mu} = (\mu_k)_{k\in B}$ that solves a system of equations (4) (given P). The entries of vector $\boldsymbol{\mu}$ are marginal rates of substitution between the expected reward from winning the prize and the cost of effort at the individual battlefields. Given a battlefield $k \in B$, this is equal to

$$\mu_i^k = \frac{\partial(p_i^k(\boldsymbol{e}_1, \boldsymbol{e}_2)v^k)}{\partial e_i^k} \Bigg/ \frac{\partial(c_i e_i^k)}{\partial e_i^k} = \frac{\partial(p_i^k(\boldsymbol{e}_1, \boldsymbol{e}_2)v^k)}{\partial(c_i e_i^k)} = \frac{p_i^k(\boldsymbol{e}_1, \boldsymbol{e}_2)(1 - p_i^k(\boldsymbol{e}_1, \boldsymbol{e}_2))}{y_i^k c_i} v^k.$$

The total effort of player i is equal to the sum of values of battlefields times the scaling factor depending on the costs of efforts facing the players. This is independent of the network of spillovers. Similarly, the equilibrium probabilities of winning each battlefield as well as equilibrium payoffs are independent of the network of spillovers. This independence holds even in the case of corner equilibria.

To illustrate the workings of Theorem 1, we consider the two examples on regular network and star network. Consider a regular network $\boldsymbol{\rho}$ and strategy profile $(\boldsymbol{e}_1, \boldsymbol{e}_2)$, as defined in Example 1. Let $\boldsymbol{\mu} = (1/(1+d))\mathbf{1}$. It is elementary to verify that $(\boldsymbol{e}_1, \boldsymbol{e}_2)$

and $\boldsymbol{\mu}$ satisfy the conditions (3) and (4) of Theorem 1. By Theorem 1, $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ is the unique Nash equilibrium of the game played on the star network.

Consider a regular network $\boldsymbol{\rho}$ and strategy profile $(\boldsymbol{e}_1, \boldsymbol{e}_2)$, as defined in Example 2. Let $\boldsymbol{\mu} = (\mu^k)_{k \in B}$ with

$$\mu^k = \begin{cases} \frac{1-\sum_{j=1}^m \lambda_j}{n-m}, & \text{if } k = 0, \\ 1, & \text{if } 1 \leq k \leq m, \\ \frac{1-\sum_{j=1}^m \lambda_j}{\lambda_k(n-m)}, & \text{otherwise.} \end{cases}$$

It is elementary to verify that $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ and $\boldsymbol{\mu}$ satisfy the conditions (3) and (4) of Theorem 1. By Theorem 1, $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ is the unique Nash equilibrium of the game played on the star network.

3.3 Scope of the network invariance result

An intriguing feature of the equilibrium characterized in Theorem 1 is the network invariance result. Is this a feature that holds more generally, beyond the Tullock CSFs? Which features of the Tullock CSF drive this result? We address these questions in this section.

A contest success function $p : \mathbb{R}^2_{\geq 0} \to [0,1]^2$ satisfies *homogeneity of degree* 0 if for all $(y_1, y_2) \in \mathbb{R}^2_{\geq 0}$, all $\theta > 0$, and all $i \in \{1,2\}$, $p_i(\theta y_1, \theta y_2) = p_i(y_1, y_2)$. A contest success function satisfies the *no-tie* property if for all $(y_1, y_2) \in \mathbb{R}^2_{\geq 0}$, $p_1(y_1, y_2) + p_2(y_1, y_2) = 1$. Both conditions are satisfied by Tullock CSFs. Given a bivariate function $f(x_1, x_2)$, $D_i f = \partial f(x_1, x_2)/\partial x_i$ denotes the derivative of f with respect to the i'th argument. We now state the network invariance result (the proof is given in the Appendix).

Proposition 1. *Let p be a contest success function that is homogeneous of degree 0 and satisfies the no-tie property. At any pure strategy equilibrium $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ of the game, for any player $i \in \{1,2\}$, the equilibrium payoff to i is*

$$\Pi_i(\boldsymbol{e}_1, \boldsymbol{e}_2) = (p_i(c_2, c_1) - c_{-i} D_i p_i(c_2, c_1))\, \mathbf{1}^T \boldsymbol{v}.$$

In addition, at any interior Nash equilibrium $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ of the game, the following holds, for any player $i \in \{1,2\}$:

(i) The winning probability of i at any battlefield $k \in B$ is equal to $p_i(c_2, c_1)$.

(ii) The total expenditure of i is

$$c_i \mathbf{1}^T \boldsymbol{e}_i = c_{-i} D_i p_i(c_2, c_1)\, \mathbf{1}^T \boldsymbol{v}.$$

These objectives depend only on the contest success function, the cost ratio c_1/c_2 and the aggregate prize. In particular, they do not depend on $\boldsymbol{\rho}$.

The values of $p_i(c_2, c_1)$ and $D_i p_i(c_2, c_1)$ depend on the contest technology. When the contest success function p_i is the Tullock CSF with parameter $\gamma \in (0, 1]$, we can explicitly compute these values.

Homogeneity of degree 0 is crucial for Proposition 1 to hold. In the following example we illustrate that the network invariance result may fail to hold when the CSF does not satisfy this property.

Example 3. *Consider a more general form of CSF:*

$$p_i(y_1, y_2) = \frac{f(y_i)}{f(y_1) + f(y_2)}, \tag{5}$$

where $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing on $\mathbb{R}_{\geq 0}$ and $f(0) \geq 0$.

In this case the probability of winning battlefield k by player i satisfies

$$\frac{\partial p_i^k}{\partial y_i^k} = \frac{f'(y_i^k) f(y_j^k)}{(f(y_1) + f(y_2))^2} = p_i^k(1 - p_i^k)\frac{f'(y_i^k)}{f(y_i^k)} = \frac{p_i^k(1 - p_i^k)}{h(y_i^k)}, \tag{6}$$

where $h(y_i^k) = \frac{f(y_i^k)}{f'(y_i^k)}$.

With the CSF given in (5) the FOC wrt e_i^k in the case of interior equilbria is

$$0 = \frac{\partial \Pi_i}{\partial e_i^k} = \frac{p_i^k(1 - p_i^k)}{h(y_i^k)} v^k + \sum_l \rho_{kl} \frac{p_i^l(1 - p_i^l)}{h(y_i^l)} v^l - c_i \ , \ i \in N, \ k \in B,$$

Substituting

$$\mu_i^k := \frac{p_i^k(1 - p_i^k)}{h(y_i^k) c_i} v^k$$

and using the fact that $c_i > 0$, the system can be rewritten as

$$\mu_i^k + \sum_l \rho_{kl} \mu_i^l = 1 \ , \ i \in N, \ k \in B, \tag{7}$$

By (7), taking the ratio and using the fact that $1 - p_1^k = p_2^k$, we get

$$\frac{h(y_j^k)}{h(y_i^k)} = \frac{c_i}{c_j}. \tag{8}$$

Since

$$p_i^k = \frac{1}{1 + \frac{f(y_j^k)}{f(y_i^k)}}$$

and

$$\frac{h(y_j^k)}{h(y_i^k)} = \frac{f(y_j^k)}{f(y_i^k)} \frac{f'(y_i^k)}{f'(y_j^k)}$$

, so

$$p_1^{k*} = \frac{c_2}{c_1 \frac{f'(y_2^k)}{f'(y_1^k)} + c_2}, \; p_2^{k*} = \frac{c_1}{c_1 + c_2 \frac{f'(y_1^k)}{f'(y_2^k)}}.$$

In equilibrium, the conflict expenditures are,

$$EP_1^* = c_1 \mathbf{1}^T \boldsymbol{e}_1 = c_1 \underbrace{\mathbf{1}^T (I + \rho^T)^{-1}}_{= \boldsymbol{\mu_1^T}} \boldsymbol{y_1} = c_1 c_2 \sum_{k \in B} \frac{v^k}{\left(c_1 \frac{f'(y_2^k)}{f'(y_1^k)} + c_2\right) \left(c_1 + c_2 \frac{f'(y_1^k)}{f'(y_2^k)}\right)} \frac{y_1^k f'(y_1^k)}{f(y_1^k)},$$

$$EP_2^* = c_1 c_2 \sum_{k \in B} \frac{v^k}{\left(c_1 \frac{f'(y_2^k)}{f'(y_1^k)} + c_2\right) \left(c_1 + c_2 \frac{f'(y_1^k)}{f'(y_2^k)}\right)} \frac{y_2^k f'(y_2^k)}{f(y_2^k)}.$$

This could depend on the network, depending on the form of f. Let $f(z) = az + b$, where $a > 0$ and $b \geq 0$. In this case

$$h(y_i^k) = y_i^k + \frac{b}{a}$$

and, for any network,

$$p_1^* = \frac{c_2}{c_1 + c_2} \text{ and } p_2^* = \frac{c_1}{c_1 + c_2}$$

and are independent of the network (it does not have to be so for a general f). For any internal equilibrium, the equilibrium expenditures are equal to

$$EP_1^* = \frac{c_1 c_2}{(c_1 + c_2)^2} \sum_{k \in B} \frac{v^k a y_1^k}{a y_1^k + b},$$

$$EP_2^* = \frac{c_1 c_2}{(c_1 + c_2)^2} \sum_{k \in B} \frac{v^k a y_2^k}{a y_2^k + b}.$$

Consider a network with two battlefields, $B = \{0, 1\}$, $\rho_{10} = 0$ and $\rho_{01} = \lambda \in [0, 1/2)$ (c.f. Figure 3).[1] *The value of each battle is set to 1.*

In this case the system of equations (7) takes the form

$$\mu_i^0 + \lambda \mu_i^1 = 1$$
$$\mu_i^1 = 1.$$

[1]The condition on λ is necessary to have an interior equilibrium.

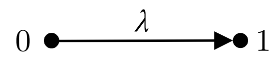

Figure 3: Dyad with one-way spillover

Which yields $\mu_i^0 = 1 - \lambda$ *and* $\mu_i^1 = 1$. *Using* (8) *and solving for* y_1^k *we obtain*

$$
\begin{aligned}
y_1^0 &= \frac{c_2}{(1-\lambda)(c_1+c_2)^2} - \frac{b}{a} \\
y_1^1 &= \frac{c_2}{(c_1+c_2)^2} - \frac{b}{a}
\end{aligned}
$$

and further, from

$$
\begin{aligned}
y_1^0 &= e_1^0 \\
y_1^1 &= \lambda e_1^0 + e_1^1,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
e_1^0 &= \frac{c_2}{(1-\lambda)(c_1+c_2)^2} - \frac{b}{a}, \\
e_1^1 &= \left(\frac{1-2\lambda}{1-\lambda}\right)\frac{c_2}{(c_1+c_2)^2} - (1-\lambda)\frac{b}{a}.
\end{aligned}
$$

By analogous derivation, in the case of player 2 *we obtain*

$$
\begin{aligned}
e_2^0 &= \frac{c_1}{(1-\lambda)(c_1+c_2)^2} - \frac{b}{a}, \\
e_2^1 &= \left(\frac{1-2\lambda}{1-\lambda}\right)\frac{c_1}{(c_1+c_2)^2} - (1-\lambda)\frac{b}{a}.
\end{aligned}
$$

Assume

$$
\frac{b}{a} < \frac{\min(c_1,c_2)(1-2\lambda)}{(c_1+c_2)^2(1-\lambda)^2}.
$$

Then $e_i^k > 0$, *for all* $k \in \{0,1\}$ *and* $i \in \{1,2\}$, *and we have an interior equilibrium. Equilibrium expenditures of players* 1 *and* 2 *are*

$$
\begin{aligned}
EP_1^* &= \frac{2c_1c_2}{(c_1+c_2)^2} - \frac{b(2-\lambda)}{a}c_1, \\
EP_2^* &= \frac{2c_1c_2}{(c_1+c_2)^2} - \frac{b(2-\lambda)}{a}c_2.
\end{aligned}
$$

The higher the spillovers (λ), *the higher the expected equilibrium expenditure of a player (as long as* $b > 0$*).*

3.4 A network index for understanding equilibrium allocations

Although the probabilities of winning a battlefield, the equilibrium total efforts and the equilibrium payoffs are network independent, the equilibrium efforts at the individual battlefields as well as the marginal rates of substitution between the expected reward from winning the prize and the cost of effort at the individual battlefields depend on the network of spillovers.

Theorem 1 ties equilibrium efforts of the players to a vector of values assigned to the nodes in the network, a *vertex index*, that reflects importance of each node in the context of contest between the two players. In this section we characterize properties of this vertex index and argue that it can be considered as a centrality index. We start with providing a characterization of this index in terms of an auxiliary optimization problem associated with the network.

Proposition 2. *Let p be the Tullock CSF with $\gamma \in (0,1]$. Strategy profile $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ is a Nash equilibrium if and only if, for any $i \in \{1,2\}$,*

$$\boldsymbol{e}_i = \frac{\gamma\,(c_1 c_2)^{\gamma}}{c_i\,(c_1^{\gamma} + c_2^{\gamma})^2}\boldsymbol{x},$$

where

$$\begin{aligned} \boldsymbol{x} \in \arg\min_{\boldsymbol{z}\in\mathbb{R}^B}\ & \mathbf{1}^T\boldsymbol{z}\ \ s.t. \\ & (\mathbf{I}+\boldsymbol{\rho})\mathrm{diag}(\boldsymbol{v})\left(\mathbf{1}\oslash\left(\left(\mathbf{I}+\boldsymbol{\rho}^T\right)\boldsymbol{z}\right)\right) \leq \mathbf{1} \\ & \boldsymbol{z} \geq \mathbf{0}. \end{aligned} \tag{9}$$

By Theorem 1 and Proposition 2, there is a unique vertex index, $\boldsymbol{x}$, solving the optimization problem (9). This index, together with the costs, determines equilibrium efforts of the players. We use the optimization problem to obtain sufficient conditions for a battlefield to receive zero or non-zero efforts from the players, as well as sufficient conditions for a node to receive higher efforts than other nodes. Before stating the result, we recall graph theoretic notions of an out-dominating set and a source.

A set of vertices, $B' \subseteq B$ is an *out-dominating set* in graph $\boldsymbol{\rho}$ if and only if, for any $i \in B \setminus B'$ there exists $j \in B'$ such that $\rho_{ji} > 0$ (c.f. Chartrand et al. (1999), for example). Informally speaking, a set of battlefields is out-dominating if and only if every battlefield that is not in the set receives a positive spillover from a node in the set. A vertex $i \in B$ is a *source* if and only if for all $j \in B$, $\rho_{ji} = 0$. Thus a battlefield that does not receive any positive spillover from another battlefield is a source.

We also introduce the following notion of domination between the nodes. Node $i \in B$ *dominates* node $j \in B$ under graph $\boldsymbol{\rho}$ if and only if for all $k \in B \setminus \{i,j\}$, $\rho_{ik} \geq \rho_{jk}$, and $\rho_{ij} \geq 1 \geq \rho_{ji}$. Thus i dominates j if it has at least as high spillovers to all the battlefields (other than i and j) as j. Node $i \in B$ *strictly dominates* node $j \in B$ if and only if $\rho_{ij} > 1 > \rho_{ji}$ and, for all $k \in B \setminus \{i,j\}$, either $\rho_{jk} = 0$ or $\rho_{ik} > \rho_{jk}$. Thus i dominates j if it has strictly higher spillovers to all the battlefields to which j has positive spillovers, spillover from i to j is greater than 1, and spillover from j to i is less than 1. We are now ready to state the result.

Proposition 3. *Let $\boldsymbol{x}$ be a vector solving (9). Then $P(\boldsymbol{x}) = \{k \in B : x_k > 0\}$ is an out-dominating set in $\boldsymbol{\rho}$. Moreover, the following hold for any $i \in B$:*

1. *If i is a source in $\boldsymbol{\rho}$ then $i \in P(\boldsymbol{x})$.*
2. *For any $j \in B$ such that j dominates i, $x_j \geq x_i$.*
3. *If there exists $j \in B$ that strictly dominates i then $x_i = 0$.*

The relation of domination is a partial order on the set of nodes. It could be viewed as a generalization, to the weighted directed graphs, of the notion of domination related to *neighbourhood inclusion* introduced by Schoch and Brandes (2016) for unweighted and undirected graphs. In the directed weighted graphs, if node i dominates node j then out-neighbourhood or i (the set of nodes to which positive weight links outgoing from i point to) is a superset of the out-neighbourhood of j and for each node k in the out-neighbourhood of j, the link from i to k has at least as high weight as the link from j to k. Schoch and Brandes (2016) argue that preserving neighbourhood inclusion is a necessary and sufficient condition for a vertex index to be considered as a centrality index. Point 2 of Proposition 3 states that the vertex index associated with equilibrium efforts of the game in question preserves out-neighbourhood inclusion. Hence it can be considered a centrality index with respect to the outgoing links.

Point 1 of Proposition 3 gives sufficient conditions for a node to have positive index values and point 3 of Proposition 3 gives sufficient conditions for a node to have zero index value (and, consequently, to receive zero efforts from the two players in equilibrium).

To illustrate the network index and its relation to the equilibrium efforts of the players, consider Examples 1 and 2. In both examples the network index is represented by vector $\boldsymbol{x}$. In the case of Example 1, with the regular network, $\boldsymbol{x} = \mathbf{1}$ and each player assigns equal effort to each battlefield. In the case of Example 2, with the star network, the nodes can be divided into three categories: the centre, the low spillovers periphery nodes (the first m periphery nodes, with sufficiently low spillovers from the centre) and the high spillovers periphery nodes (the remaining $n-m-1$ periphery nodes). The network index of the high spillovers periphery nodes is zero. These nodes do not receive any direct effort from any player in equilibrium. The network index of a low spillover periphery node is:

$$1 \; - \; (\,network\ index\ of\ the\ centre) \times (spillover\ to\ the\ periphery\ node). \tag{10}$$

In equilibrium, these nodes receive positive direct effort from both player. The lower the spillover, the higher the efforts of the players. The index of the centre is:

$$\frac{\#\ high\ spillovers\ periphery\ nodes\ +\ 1}{1\ -\ total\ spillovers\ to\ high\ spillovers\ periphery\ nodes}. \tag{11}$$

By the definition of the cutoff index, m, the total spillovers are less then one. In particular, if the lowest spillover to a periphery node is at least 1 then $m = 0$ and all the periphery nodes have high spillovers. Thus both players always allocate positive direct effort to the center while all, some or even none of the periphery nodes receive effort in equilibrium, depending on the spillovers from the centre to them. The effort directed to the centre is increasing in the spillovers to the low spillovers periphery nodes, as long as the number of low spillovers periphery nodes remains unchanged.

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A Appendix

A.1 Equilibrium characterization

In this section we prove Theorem 1, characterizing Nash equilibria in the model. The proof consists of a number of steps, stated as separate results. First, we state and prove Proposition 4 on existence of Nash equilibria. Second, we prove an auxiliary lemma which leads to a statement and a proof Proposition 5 on uniqueness of Nash equilibria. Lastly, we prove the theorem.

We start with stating and proving the existence result. The key challenge to obtain the result is the discontinuity of payoffs when neither player exerts positive efforts at a battle. To address this challenge, we employ an approximation approach by introducing a sequence of truncated contest models with a diminishing small yet positive minimum threshold on the efforts, and then taking the limit. This is a standard method of establishing existence in contest models. However, we must also consider the complicating factor of spillovers across battlefields, which can make the discontinuity in payoffs more complex. By carefully accounting for these spillovers and applying the approximation approach, we are able to rigorously verify that the limit exists and constitutes an equilibrium of the original (untruncated) game.

Proposition 4. *If p is the Tullock CSF with $\gamma \in (0,1]$, for all $i \in \{1,2\}$, then there exists a unique pure strategy Nash equilibrium.*

Proof. Let $\mathit{\Gamma}$ denote the original conflict game. For any positive $\varepsilon > 0$ we define a truncated conflict game $\mathit{\Gamma}^{\varepsilon}$ in which each player's effort in any battle is bounded below by ε. In the truncated game $\mathit{\Gamma}^{\varepsilon}$, each i's payoff, as defined in (1), is continuously differentiable. Moreover, for any $i \in \{1,2\}$, $\mathit{\Pi}_i$ is concave in $\boldsymbol{e}_i$, as $\boldsymbol{y}_i$ results from an affine transformation of $\boldsymbol{e}_i$, the CSF p_i is concave in y_i^k (as p_i is the Tullock CSF with $\gamma \in (0,1]$), and the cost is convex in $\boldsymbol{e}_i$. In addition, since the total prize is bounded from above by $\sum_{k\in B} v^k$ and the winning probabilities at every battlefield are bounded from above by 1, there is an upper bound M on the efforts so that exerting an effort higher than M on any battlefield is strictly dominated. Hence, without loss of generality, we can assume that each player's strategy space is $[\varepsilon, M]^B$. If $\varepsilon < M$ then each player's strategy space is convex, compact, and non empty. Therefore, for any $\varepsilon \in (0, M)$, by Glicksberg's fixed point theorem (Glicksberg (1952)), a pure strategy Nash equilibrium of $\mathit{\Gamma}^{\varepsilon}$ exists

For each positive integer value $l = 1, 2, \ldots$ let $\boldsymbol{e}^*(l) = (\boldsymbol{e}_1^*(l), \boldsymbol{e}_2^*(l))$ denote an equilibrium of $\Gamma^{1/(q+l)}$ where $q > 1/M$ is an integer. Since the sequence $(\boldsymbol{e}^*(l))_{l=1}^{+\infty}$ lies in the compact set $[0, M]^B \times [0, M]^B$, it has a convergent subsequence. To simplify the notation, we may assume that the sequence $\boldsymbol{e}^*(l)$ itself converges to a limit $\boldsymbol{e}^*(+\infty)$.

Let $\boldsymbol{y}_i^*(l) = (\mathbf{I} + \boldsymbol{\rho}^T)\boldsymbol{e}_i^*(l)$ be the effective efforts of player $i \in \{1, 2\}$ associated with the true efforts $\boldsymbol{e}^*(l)$, for $l \in \mathbb{N} \cup \{+\infty\}$. We will show that $\boldsymbol{e}^*(+\infty)$ is an equilibrium of the original game. To this end, we will show that:

(i) for any battle $k \in B$ there exists a player $i \in \{1, 2\}$ such that $y_i^{*k}(+\infty) > 0$, and

(ii) $\boldsymbol{e}^*(+\infty)$ is indeed an equilibrium.

For point (i) assume, to the contrary, that there exists a battlefield $k \in B$ such that for all $i \in \{1, 2\}$, $y_i^{k*}(+\infty) = 0$.

Let

$$p_i^{k*}(l) = \frac{(y_i^{k*}(l))^\gamma}{(y_1^{k*}(l))^\gamma + (y_2^{k*}(l))^\gamma}$$

be player i's winning probability at battlefield k in $\Gamma^{1/l}$ under strategy profile $\boldsymbol{e}^*(l)$. Since $p_1^{k*}(l) + p_2^{k*}(l) = 1$ so there exists a player $j \in \{1, 2\}$ such that $p_j^{k*}(l) \leq 1/2$ for infinitely many $l \in \mathbb{N}$. Taking a subsequence if necessary, we can assume that $p_j^{k*}(l) \leq 1/2$ for any $l \in \mathbb{N}$.

Now consider

$$\begin{aligned}
\left.\frac{\partial \Pi_j(\boldsymbol{e})}{\partial e_j^k}\right|_{\boldsymbol{e}=\boldsymbol{e}^*(l)} &\geq -c_j + v^k \left.\frac{\partial p_j^k(\boldsymbol{e})}{\partial e_j^k}\right|_{\boldsymbol{e}=\boldsymbol{e}^*(l)} \\
&= -c_j + v^k \left(1 - \frac{(y_j^{k*}(l))^\gamma}{(y_1^{k*}(l))^\gamma + (y_2^{k*}(l))^\gamma}\right) \frac{1}{(y_1^{k*}(l))^\gamma + (y_2^{k*}(l))^\gamma} \\
&\geq -c_j + v^k \left(1 - \frac{1}{2}\right) \frac{1}{(y_1^{k*}(l))^\gamma + (y_2^{k*}(l))^\gamma},
\end{aligned}$$

where, in the first step, we ignore the nonnegative spillovers of e_j^k on other battlefields $k \in B$, the second step follows from direct computation, and the last step follows from the fact that $p_j^{k*}(l) \leq 1/2$. Since $y_i^{k*}(+\infty) = 0$, for all $i \in N$, we have $\lim_{l \to +\infty} \left.\frac{\partial \Pi_j(\boldsymbol{e})}{\partial e_j^k}\right|_{\boldsymbol{e}=\boldsymbol{e}^*(l)} = +\infty$, as $\lim_{l \to +\infty} y_i^{k*}(l) = y_i^{k*}(+\infty) = 0$, for all $i \in \{1, 2\}$. Note that $\lim_{l \to +\infty} e_j^{k*}(l) \leq \lim_{l \to +\infty} y_j^{k*}(l) = 0$, implying that $\lim_{l \to +\infty} e_j^{k*}(l) = 0$.

Consequently, for sufficiently large l, $\left.\frac{\partial \Pi_j(\boldsymbol{e})}{\partial e_j^k}\right|_{\boldsymbol{e}=\boldsymbol{e}^*(l)} > 0$ and $e_j^{k*}(l) < M$, implying that j can strictly improve his payoff by slightly increasing his effort in battle k, which contradicts the fact that $\boldsymbol{e}^*(l)$ is an equilibrium of $\Gamma^{1/l}$.

For point (ii), take any battle $k \in B$ and any player $i \in \{1,2\}$. We need to show that for all $\boldsymbol{e}_i^* \in \mathbb{R}^B$,

$$\Pi_i(\boldsymbol{e}_1^*(+\infty), \boldsymbol{e}_2^*(+\infty)) \geq \Pi_i(\boldsymbol{e}_i, \boldsymbol{e}_{-i}^*(+\infty)).$$

Since $\boldsymbol{e}^*(l)$ is an equilibrium of $\Gamma^{1/l}$, so, for any $i \in \{1,2\}$ and any $\boldsymbol{e}_i \in [1/l, M]^B \subseteq \mathbb{R}^B$,

$$\Pi_i\left(\boldsymbol{e}_1^*(l), \boldsymbol{e}_2^*(l)\right) \geq \Pi_i\left(\boldsymbol{e}_i, \boldsymbol{e}_{-i}^*(l)\right). \tag{12}$$

Take any player $i \in \{1,2\}$ and any $\boldsymbol{e}_i \in \mathbb{R}^B$, there are two cases to consider:

Case 1: $\boldsymbol{e}_i > \boldsymbol{0}$ (every entry is positive). Then, for sufficiently large l, $\boldsymbol{e}_i \in [1/l, M]^B$. When l goes to infinity in (12) we get

$$\Pi_i(\boldsymbol{e}_1^*(+\infty), \boldsymbol{e}_2^*(+\infty)) \geq \Pi_i(\boldsymbol{e}_i, \boldsymbol{e}_{-i}^*(+\infty)),$$

as, when $l \to +\infty$, $\boldsymbol{e}^*(l) \to \boldsymbol{e}^*(+\infty)$, $\Pi_i(\boldsymbol{e}_1^*(l), \boldsymbol{e}_2^*(l)) \to \Pi_i(\boldsymbol{e}_i^*(+\infty), \boldsymbol{e}_2^*(+\infty))$ (due to the continuity of Π_i at $\boldsymbol{e} = \boldsymbol{e}^*(+\infty)$ by point (i)), and $\Pi_i(\boldsymbol{e}_1, \boldsymbol{e}_2^*(l)) \to \Pi_i(\boldsymbol{e}_i, \boldsymbol{e}_{-i}^*(+\infty))$ (due to the continuity of Π_i at $\boldsymbol{e} = (\boldsymbol{e}_i, \boldsymbol{e}_{-i}^*(+\infty))$).[2]

Case 2: $\boldsymbol{e}_i \geq \boldsymbol{0}$. Take any $\eta > 0$, and consider $\hat{\boldsymbol{e}}_i = \eta\boldsymbol{1} + \boldsymbol{e}_i > \boldsymbol{0}$ (where $\boldsymbol{1}$ is the vector of 1's). Then,

$$\Pi_i(\boldsymbol{e}_1^*(+\infty), \boldsymbol{e}_2^*(+\infty)) \geq \Pi_i(\hat{\boldsymbol{e}}_i, \boldsymbol{e}_{-i}^*(+\infty)),$$

by Case 1.

Furthermore, the winning probability of i weakly increases when i's efforts increases from $\boldsymbol{e}_i$ to $\hat{\boldsymbol{e}}_i$. The cost difference between $\hat{\boldsymbol{e}}_i$ and $\boldsymbol{e}_i$ is $c_i m\eta$. Therefore,

$$\Pi_i(\hat{\boldsymbol{e}}_i, \boldsymbol{e}_{-i}^*(+\infty)) \geq \Pi_i(\boldsymbol{e}_i, \boldsymbol{e}_{-i}^*(+\infty)) - c_i m\eta.$$

Combining these inequalities yields

$$\Pi_i(\boldsymbol{e}_i^*(+\infty), \boldsymbol{e}_{-i}^*(+\infty)) \geq \Pi_i(\boldsymbol{e}_i, \boldsymbol{e}_{-i}^*(+\infty)) - c_i m\eta,$$

which holds for any $\eta > 0$. Taking $\eta \to 0^+$ yields $\Pi_i(\boldsymbol{e}_i^*(+\infty), \boldsymbol{e}_{-i}^*(+\infty)) \geq \Pi_i(\boldsymbol{e}_i, \boldsymbol{e}_{-i}^*(+\infty))$. $\square$

[2] For any strategy profile $\boldsymbol{e} \in \mathbb{R}_{\geq 0}^B \times \mathbb{R}_{\geq 0}^B$, as long as for any battlefield there is one player with strictly positive effective effort, Π_i is continuous at $\boldsymbol{e}$.

Having proven existence, we move to characterization of equilibria when matrix $\mathbf{I}+\boldsymbol{\rho}$ is non-singular. In the analysis we use the following types of strategy profiles, adopted from Xu et al. (2022). A strategy profile, $(\boldsymbol{e}_1, \boldsymbol{e}_2)$, with the associated profile of effective effort, $(\boldsymbol{y}_1, \boldsymbol{y}_2)$, is of type S^1 if for each battlefield $k \in B$ there exists player $i \in \{1, 2\}$ such that $y_i^k > 0$. A strategy profile is of type S^2 if for each battlefield $k \in B$ and each player $i \in \{1, 2\}$, $y_i^k > 0$. The distinction between these two types of strategy profiles follows Xu et al. (2022), but is based on the effective efforts, $(\boldsymbol{y}_1, \boldsymbol{y}_2)$, rather than the true efforts, $(\boldsymbol{e}_1, \boldsymbol{e}_2)$. We will also use the standard notion of exchangeability of equilibria, defined as follows. The set of Nash equilibria is exchangeable if for any two Nash equilibria $(\boldsymbol{e}'_1, \boldsymbol{e}'_2)$ and $(\boldsymbol{e}''_1, \boldsymbol{e}''_2)$, the strategy profiles $(\boldsymbol{e}'_1, \boldsymbol{e}''_2)$ and $(\boldsymbol{e}''_1, \boldsymbol{e}'_2)$ are also Nash equilibria.

We start with the following auxiliary lemma.

Lemma 1. *If p_i is the Tullock CSF with $\gamma \in (0, 1]$, for all $i \in \{1, 2\}$ then*

(i) The set of equilibria is convex.

(ii) If there exists a type S^2 equilibrium then it must be the unique equilibrium.

(iii) The set of equilibria is exchangeable.

Proof. If matrix $\mathbf{I}+\boldsymbol{\rho}$ is non-singular, the linear mapping $\boldsymbol{e}_i \to \left(\mathbf{I}+\boldsymbol{\rho}^T\right)\boldsymbol{e}_i$ is invertible and one-to-one. Let

$$Y = \{\boldsymbol{y} = \left(\mathbf{I}+\boldsymbol{\rho}^T\right)\boldsymbol{e}, \text{ where } \boldsymbol{e} \in \mathbb{R}^B_{\geq 0}\}$$

be the image of the first quadrant $\mathbb{R}^m_{\geq 0}$ under this linear mapping. Clearly, Y is convex and closed. If we reformulate the conflict game using effective efforts $\boldsymbol{y}_i \in Y$ instead of true efforts $\boldsymbol{e}_i \in \mathbb{R}^B_{\geq 0}$ as the strategies of each player $i \in \{1, 2\}$, the efforts are independent across the battlefields. However, we need to redefine the cost function and take into account the interdepedence between the effective efforts. The payoff to player i from the (effective) effort profile $(\boldsymbol{y}_1, \boldsymbol{y}_2) \in Y^2$ of this conflict game is

$$\begin{aligned} \widetilde{\Pi}_i(\boldsymbol{y}_1, \boldsymbol{y}_2) &= \Pi_i\left(\left(\mathbf{I}+\boldsymbol{\rho}^T\right)^{-1}\boldsymbol{y}_1, \left(\mathbf{I}+\boldsymbol{\rho}^T\right)^{-1}\boldsymbol{y}_2\right) \\ &= \sum_{k\in B} v^k p_i(y_1^k, y_2^k) - c_i \mathbf{1}^T\left(\mathbf{I}+\boldsymbol{\rho}^T\right)^{-1}\boldsymbol{y}_i, \end{aligned} \tag{13}$$

Since mapping $\mathbf{I}+\boldsymbol{\rho}$ is linear and invertible, a strategy profile $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ is a Nash equilibrium of the original game if and only if the associated profile of effective efforts, $(\boldsymbol{y}_1, \boldsymbol{y}_2)$, is a Nash equilibrium in the game in the space of effective efforts. In the reformulated conflict game, using effective efforts as strategies, we can readily apply results from Xu et al. (2022). Since the mapping between the true efforts, $\mathbb{R}^B_{\geq 0}$, and the effective efforts, Y, is invertible, linear, and one-to-one, convexity/uniqueness in the space of effective efforts, Y, implies convexity/uniqueness in the space of $\boldsymbol{e}_i$. Notice that although the winning probabilities as a function of true efforts are independent across the battlefields in Xu et al. (2022), while they are interdependent in our setting with spillovers, using effective efforts $\boldsymbol{y}_i$ as the choice variables, the winning probability at each battlefield $k \in B$ depends only on the effective efforts at the battlefield and is independent of the efforts at any other battlefield $k' \neq k$. However, the strategy set Y is no longer the first quadrant $\mathbb{R}^m_{\geq 0}$, but its image under the linear transformation $\mathbf{I} + \boldsymbol{\rho}$. (Xu et al., 2022, Section 3.4.1) show how to address complex restrictions on players' strategy spaces. In particular, by (Xu et al., 2022, Proposition 4), the set of equilibria is convex. Since its image under the linear transformation $(\mathbf{I} + \boldsymbol{\rho})^{-1}$ is also convex, point (i) of the lemma follows. By (Chin et al., 1974, Theorem 1), convexity implies exchangeability. Hence point (iii) follows. Furthermore, by (Xu et al., 2022, Proposition 5), there is at most one equilibrium of type S^2, which proves point (ii) of the lemma. □

With Lemma 1 in hand, we are ready to state and prove the result about uniqueness of equilibria in the model. In addition, we also show that in equilibrium every battlefield has a positive effective efforts from every player assigned to it. It may, however, have 0 real effort assigned from both players, in which case the effort assigned to it comes purely from the spillovers from other battlefields.

Proposition 5. *If p_i is the Tullock CSF with $\gamma \in (0, 1]$, for all $i \in \{1, 2\}$, and $\mathbf{I} + \boldsymbol{\rho}$ is non-singular then Nash equilibrium is unique. Moreover, in equilibrium, every battlefield receives positive effective effort from every player $i \in \{1, 2\}$.*

Proof. Notice that if strategy profile $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ is a Nash equilibrium then it must be of type S^1. For if there is a battlefield k with $(y^k_1, y^k_2) = (0, 0)$ then $(e^k_1, e^k_2) = 0$ and increasing e^k_i by an arbitrarily small $\varepsilon > 0$ allows i to win k with probability 1 and gain $v^k > 0$ for an arbitrarily small cost εc_i. Hence i is able to deviate to a strategy that gets him a positive increase in payoff, a contradiction with the assumption that $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ is a Nash equilibrium.

First we consider the case of $c_1 = c_2 = 1$. By Proposition 4, there exists a Nash equilibrium. Suppose that strategy profile $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ with the associated effective efforts profile $(\boldsymbol{z}_1, \boldsymbol{z}_2)$ is a Nash equilibrium. By the observation above, it is of type S^1. Given the symmetry of players, $(\boldsymbol{z}_2, \boldsymbol{z}_1)$ is also a Nash equilibrium. Moreover, a strategy profile $(\boldsymbol{e}'_1, \boldsymbol{e}'_2) = ((\boldsymbol{e}_1 + \boldsymbol{e}_2)/2, (\boldsymbol{e}_1 + \boldsymbol{e}_2)/2)$ with the associated effective efforts profile $((\boldsymbol{z}_1+\boldsymbol{z}_2)/2, (\boldsymbol{z}_1+\boldsymbol{z}_2)/2)$, is also an equilibrium by point (i) of Lemma 1.

Since both $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ and $(\boldsymbol{e}_2, \boldsymbol{e}_1)$ are of type S^1, $(\boldsymbol{e}'_1, \boldsymbol{e}'_2)$ is of type S^2. Hence, by point (ii) of Lemma 1, $(\boldsymbol{e}'_1, \boldsymbol{e}'_2)$ is the unique equilibrium.

Thus we have established that if $c_1 = c_2 = 1$ then there exits a unique Nash equilibrium, which is symmetric and of type S^2. Let $(\hat{\boldsymbol{e}}, \hat{\boldsymbol{e}})$ denote the unique equilibrium when $c_1 = c_2 = 1$. Since this equilibrium is of type S^2, $\left(\mathbf{I} + \boldsymbol{\rho}^T\right) \hat{\boldsymbol{e}}$ is component-wise positive, i.e.,

$$\hat{\boldsymbol{y}} = \left(\mathbf{I} + \boldsymbol{\rho}^T\right) \hat{\boldsymbol{e}} \in \mathbb{R}^B_{>0}. \tag{14}$$

Next, we consider the general case of $c_1 > 0$ and $c_2 > 0$. For $i \in \{1, 2\}$, let

$$\boldsymbol{e}'_i = \frac{4c_1^\gamma c_2^\gamma}{c_i(c_1^\gamma + c_2^\gamma)^2} \hat{\boldsymbol{e}}. \tag{15}$$

By construction, the strategy profile $(\boldsymbol{e}'_1, \boldsymbol{e}'_2)$ is of type S^2, as each player's strategy is proportional to $\hat{\boldsymbol{e}}$. Next we prove that $(\boldsymbol{e}'_1, \boldsymbol{e}'_2)$ is a Nash equilibrium. Notice that a strategy profile $(\boldsymbol{e}_1, \boldsymbol{e}_2)$, of type S^1, with the associated effective efforts profile, $(\boldsymbol{y}_1, \boldsymbol{y}_2)$, is a Nash equilibrium of our game if and only if, for any for any $i \in \{1, 2\}$ and any $k \in B$ it satisfies

$$\begin{aligned}
\frac{\partial \Pi_i}{\partial e_i^k} &= \frac{\gamma p_i^k (1 - p_i^k)}{y_i^k} v^k + \sum_{l \in B \setminus \{k\}} \rho_{kl} \frac{\gamma p_i^l (1 - p_i^l)}{y_i^l} v^l - c_i = 0, \text{ if } e_i^k > 0, \\
\frac{\partial \Pi_i}{\partial e_i^k} &= \frac{\gamma p_i^k (1 - p_i^k)}{y_i^k} v^k + \sum_{l \in B \setminus \{k\}} \rho_{kl} \frac{\gamma p_i^l (1 - p_i^l)}{y_i^l} v^l - c_i \leq 0, \text{ if } e_i^k = 0,
\end{aligned} \tag{16}$$

where the winning probability p_i^k is given in (2). The "only if" direction is clear as (16) is the first order condition of player i with respect to e_i^k (the payoff of i is continuously differentiable in $\boldsymbol{e}_i$ as $\boldsymbol{e}$ is of type S^1). The "if" direction follows because the payoff function of player i is concave and the set of admissible efforts is convex. Therefore the local optimality condition given in (16) implies global optimality.

In the case of $c_1 = c_2 = 1$, at $(\hat{\boldsymbol{e}}, \hat{\boldsymbol{e}})$, (16) reduces to

$$\begin{aligned}
&\frac{\gamma v^k}{4\hat{y}^k} + \sum_{l \in B \setminus \{k\}} \rho_{kl} \frac{\gamma v^l}{4\hat{y}^l} - 1 = 0, \text{ if } \hat{e}^k > 0 \\
&\frac{\gamma v^k}{4\hat{y}^k} + \sum_{l \in B \setminus \{k\}} \rho_{kl} \frac{\gamma v^l}{4\hat{y}^l} - 1 \leq 0, \text{ if } \hat{e}^k = 0,
\end{aligned} \tag{17}$$

(notice that $p_1^k = p_2^k = 1/2$ in this case, because the efforts of the two players at each battlefield are equal). Since $(\hat{\boldsymbol{e}}, \hat{\boldsymbol{e}})$ is a Nash equilibrium when $c_1 = c_2 = 1$, (17) is satisfied for all $i \in \{1, 2\}$ and all $k \in B$.

In the general case general $c_1 > 0$ and $c_2 > 0$, at $(\boldsymbol{e}'_1, \boldsymbol{e}'_2)$, we have $p_1^k = c_2^\gamma/(c_1^\gamma + c_2^\gamma)$ an $p_2^k = c_1^\gamma/(c_1^\gamma + c_2^\gamma)$. Therefore, for $i \in \{1, 2\}$,

$$\frac{p_i^k(1 - p_i^k)}{{y'}_i^k} = \frac{c_1^\gamma c_2^\gamma}{(c_1^\gamma + c_2^\gamma)^2} \bigg/ \frac{4c_1^\gamma c_2^\gamma}{c_i(c_1^\gamma + c_2^\gamma)^2}\hat{y}^k = \frac{c_i}{4\hat{y}^k} \tag{18}$$

and (16) can be rewritten as

$$\begin{aligned} c_i\frac{\gamma v^k}{4\hat{y}^k} + c_i \sum_{l \in B\setminus\{k\}} \rho_{kl}\frac{\gamma v^l}{4\hat{y}^l} - c_i = 0, \text{ if } \hat{e}^k > 0,\\ c_i\frac{\gamma v^k}{4\hat{y}^k} + c_i \sum_{l \in B\setminus\{k\}} \rho_{kl}\frac{\gamma v^l}{4\hat{y}^l} - c_i \leq 0, \text{ if } \hat{e}^k = 0. \end{aligned} \tag{19}$$

Since $c_i > 0$, (19) holds by (17). Therefore, $(\boldsymbol{e}'_1, \boldsymbol{e}'_2)$ is indeed an equibrium at costs c_1 and c_2. Recall that strategy profile $(\boldsymbol{e}'_1, \boldsymbol{e}'_2)$ is of type S^2. Hence, by point (ii) of Lemma 1, it is the unique Nash equilibrium. $\square$

We now give a proof of Theorem 1.

Proof of Theorem 1. The existence part of the theorem follows from Proposition 4 and the uniqueness part, together with every battlefield receiving positive effective effort from each player in equilibrium, follows from Proposition 5. What remains to be shown are the explicit formulas for the equilibrium efforts as well as the formulas for equilibrium total efforts, probabilities of winning, and the payoffs. We provide them below.

As we argued in proof of Proposition 5, the strategy profile $(\boldsymbol{e}'_1, \boldsymbol{e}'_2)$ such that $\boldsymbol{e}'_i$ with $i \in \{1, 2\}$ satisfies (15) and efforts vector $\hat{\boldsymbol{e}}$ with the associated effective efforts vector $\hat{\boldsymbol{y}}$ satisfies, for any $i \in \{1, 2\}$ and $k \in B$, the system of equations and inequalities (17), is the unique Nash equilibrium of the model.

Let

$$\mu_i^k = \frac{\partial(p_i^k v^k)}{\partial e_i^k} \bigg/ \frac{\partial(c_i e_i^k)}{\partial e_i^k} = \frac{\partial(p_i^k v^k)}{\partial(c_i e_i^k)}$$

be the marginal rate of substitution between the expected reward from winning the prize and the cost of effort. At the strategy profile $(\boldsymbol{e}'_1, \boldsymbol{e}'_2)$,

$$\mu_i^k = \frac{\gamma p_i^k(1 - p_i^k)}{{y'}_i^k c_i} v^k,$$

where $p_i^k = p_i({y'}_1^k, {y'}_2^k)$. Moreover, by (18),

$$\mu_1^k = \mu_2^k = \hat{\mu}^k = \frac{\gamma v^k}{4\hat{y}^k}. \tag{20}$$

Using that, (17) can be rewritten as

$$\hat{\mu}^k + \sum_{l \in B \setminus \{k\}} \rho_{kl} \hat{\mu}^l = 1 \text{ , if } \hat{e}^k > 0,$$
$$\hat{\mu}^k + \sum_{l \in B \setminus \{k\}} \rho_{kl} \hat{\mu}^l \leq 1 \text{ , if } \hat{e}^k = 0.$$

and, further, introducing slack variables s^k (for $k \in B$), as

$$\hat{\mu}^k + \sum_{l \in B \setminus \{k\}} \rho_{kl} \hat{\mu}^l = 1 - s^k,$$

where, for all $k \in B$, $s^k \geq 0$ and $s^k = 0$, if $\hat{e}^k > 0$. In matrix form,

$$(\mathbf{I} + \boldsymbol{\rho})\, \hat{\boldsymbol{\mu}} = \mathbf{1} - \boldsymbol{s}, \implies \hat{\boldsymbol{\mu}} = (\mathbf{I} + \boldsymbol{\rho})^{-1} (\mathbf{1} - \boldsymbol{s}). \tag{21}$$

By (20), for all $k \in B$,

$$\hat{y}^k = \frac{\gamma v^k}{4 \hat{\mu}^k}.$$

In matrix form,

$$\hat{\boldsymbol{y}} = \frac{\gamma}{4} \begin{bmatrix} \frac{v^1}{\hat{\mu}^1} \\ \vdots \\ \frac{v^k}{\hat{\mu}^k} \end{bmatrix} = \frac{\gamma}{4} \boldsymbol{v} \oslash \hat{\boldsymbol{\mu}} \tag{22}$$

and, given the definition of $\hat{\boldsymbol{y}}$ in terms of $\hat{\boldsymbol{e}}$,

$$\hat{\boldsymbol{e}} = \left(\mathbf{I} + \boldsymbol{\rho}^T\right)^{-1} \hat{\boldsymbol{y}} = \frac{\gamma}{4} \left(\mathbf{I} + \boldsymbol{\rho}^T\right)^{-1} (\boldsymbol{v} \oslash \hat{\boldsymbol{\mu}}). \tag{23}$$

Let $P = \{k \in B : e'_k > 0\}$ be the set of battlefields receiving positive real effort under $\boldsymbol{e}'$ and let $-P = B \setminus \{P\}$ be the set of battlefields receiving zero real effort under $\boldsymbol{e}'$. By (21)

$$(\mathbf{I}_{P,P} + \boldsymbol{\rho}_{P,P})\, \hat{\boldsymbol{\mu}}_P + \boldsymbol{\rho}_{P,-P} \hat{\boldsymbol{\mu}}_{-P} = \mathbf{1}.$$

Moreover, since $\hat{\boldsymbol{e}}_{-P} = 0$ and $\hat{\boldsymbol{y}} = (\mathbf{I} + \boldsymbol{\rho}^T)\hat{\boldsymbol{e}}$ so

$$\hat{\boldsymbol{y}}_P = \left(\mathbf{I}_P + \boldsymbol{\rho}_{P,P}^T\right) \hat{\boldsymbol{e}}_P$$
$$\hat{\boldsymbol{y}}_{-P} = \boldsymbol{\rho}_{P,-P}^T \hat{\boldsymbol{e}}_P$$

Hence, using (22),

$$\hat{\boldsymbol{e}}_P = \frac{\gamma}{4} \left(\mathbf{I}_P + \boldsymbol{\rho}_{P,P}^T\right)^{-1} (\boldsymbol{v}_P \oslash \hat{\boldsymbol{\mu}}_P)$$
$$\boldsymbol{v}_{-P} \oslash \hat{\boldsymbol{\mu}}_{-P} = \boldsymbol{\rho}_{P,-P}^T \hat{\boldsymbol{e}}_P = \frac{\gamma}{4} \boldsymbol{\rho}_{P,-P}^T \left(\mathbf{I}_P + \boldsymbol{\rho}_{P,P}^T\right)^{-1} (\boldsymbol{v}_P \oslash \hat{\boldsymbol{\mu}}_P)$$

Thus we obtained the system of equations defining the equilibrium efforts.

As we already noticed in proof of Proposition 5, the probability of winning battlefield $k \in B$ by player $i \in \{1, 2\}$ in the unique Nash equilibrium $(\boldsymbol{e}'_1, \boldsymbol{e}'_2)$,

$$p_i^k(\boldsymbol{e}'_1, \boldsymbol{e}'_2) = \frac{c_{-i}^{\gamma}}{c_1^{\gamma} + c_2^{\gamma}}.$$

The total effort of player $i \in \{1, 2\}$ in the unique Nash equilibrium $(\boldsymbol{e}'_1, \boldsymbol{e}'_2)$ is

$$\sum_{k \in B} e_i'^k = \mathbf{1}^T \boldsymbol{e}'_i = \frac{4c_1^{\gamma} c_2^{\gamma}}{c_i(c_1^{\gamma} + c_2^{\gamma})^2} \mathbf{1}^T \hat{\boldsymbol{e}} = \frac{4c_1^{\gamma} c_2^{\gamma}}{c_i(c_1^{\gamma} + c_2^{\gamma})^2} (\mathbf{1} - \boldsymbol{s})^T \hat{\boldsymbol{e}},$$

as $s^k = 0$, if $\hat{e}^k > 0$, and $\hat{e}^k = 0$, if $s^k > 0$. Hence

$$\sum_{k \in B} e_i'^k = \frac{\gamma c_1^{\gamma} c_2^{\gamma}}{c_i(c_1^{\gamma} + c_2^{\gamma})^2} \underbrace{(\mathbf{1} - \boldsymbol{s})^T \left(\mathbf{I} + \boldsymbol{\rho}^T\right)^{-1}}_{=\hat{\boldsymbol{\mu}}^T} \boldsymbol{v} \oslash \hat{\boldsymbol{\mu}} = \frac{\gamma c_1^{\gamma} c_2^{\gamma}}{c_i(c_1^{\gamma} + c_2^{\gamma})^2} \left(\sum_{k \in B} v^k\right) = \frac{\gamma c_1^{\gamma} c_2^{\gamma}}{c_i(c_1^{\gamma} + c_2^{\gamma})^2} \mathbf{1}^T \boldsymbol{v}.$$

The equilibrium payoff to player i is

$$\Pi_i(\boldsymbol{e}'_1, \boldsymbol{e}'_2) = \sum_{k \in B} p_i^k v^k - c_i \mathbf{1}^T \boldsymbol{e}'_i = \frac{c_{-i}^{\gamma}}{c_1^{\gamma} + c_2^{\gamma}} \mathbf{1}^T \boldsymbol{v} - \frac{\gamma c_1^{\gamma} c_2^{\gamma}}{(c_1^{\gamma} + c_2^{\gamma})^2} \mathbf{1}^T \boldsymbol{v} = \frac{c_{-i}^{\gamma} \left(c_1^{\gamma} + c_2^{\gamma} - \gamma c_i^{\gamma}\right)}{\left(c_1^{\gamma} + c_2^{\gamma}\right)^2} \mathbf{1}^T \boldsymbol{v}.$$

This completes the proof. □

A.2 Network invariance

In this section we prove Proposition 1, the network invariance result.

Proof of Proposition 1. Consider the model with a general contest success function p which is homogeneous of degree 0 and satisfies the no-tie property.

Take any strategy profile $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ with the associated effective efforts profile, $(\boldsymbol{y}_1, \boldsymbol{y}_2)$. By the no-tie property of the contest success function,

$$\begin{aligned} \Pi_1(\boldsymbol{e}_1, \boldsymbol{e}_2) + \Pi_2(\boldsymbol{e}_1, \boldsymbol{e}_2) &= \sum_{k \in B} v^k \left(p_1\left(y_1^k, y_2^k\right) + p_2\left(y_1^k, y_2^k\right)\right) - c_1 \sum_{k \in B} e_1^k - c_2 \sum_{k \in B} e_2^k \\ &= \sum_{k \in B} v^k - c_1 \sum_{k \in B} e_1^k - c_2 \sum_{k \in B} e_2^k. \end{aligned} \tag{24}$$

Similarly

$$\begin{aligned} &\Pi_1\left(\frac{c_2}{c_1}\boldsymbol{e}_2, \boldsymbol{e}_2\right) + \Pi_2\left(\boldsymbol{e}_1, \frac{c_1}{c_2}\boldsymbol{e}_1\right) \\ &\qquad = \sum_{k \in B} v^k \left(p_1\left(\frac{c_2}{c_1} y_2^k, y_2^k\right) + p_2\left(y_1^k, \frac{c_1}{c_2} y_1^k\right)\right) - c_2 \sum_{k \in B} e_2^k - c_1 \sum_{k \in B} e_1^k \end{aligned}$$

and since, by homogeneity of degree 0, for any $i \in \{1,2\}$,

$$p_i\left(\frac{c_2}{c_1}y_2^k, y_2^k\right) = p_i\left(y_1^k, \frac{c_1}{c_2}y_1^k\right) = p_i\left(1, \frac{c_1}{c_2}\right)$$

so, using the no-tie property,

$$\mathit{\Pi}_1\left(\frac{c_2}{c_1}\boldsymbol{e}_2, \boldsymbol{e}_2\right) + \mathit{\Pi}_2\left(\boldsymbol{e}_1, \frac{c_1}{c_2}\boldsymbol{e}_1\right) = \sum_{k\in B} v^k - c_1\sum_{k\in B} e_1^k - c_2\sum_{k\in B} e_2^k. \tag{25}$$

Analogously,

$$\begin{aligned}
&\mathit{\Pi}_1\left(\boldsymbol{e}_1, \frac{c_1}{c_2}\boldsymbol{e}_1\right) + \mathit{\Pi}_2\left(\frac{c_2}{c_1}\boldsymbol{e}_2, \boldsymbol{e}_2\right) \\
&\quad= \sum_{k\in B} v^k\left(p_1\left(y_1^k, \frac{c_1}{c_2}y_1^k\right) + p_2\left(\frac{c_2}{c_1}y_2^k, y_2^k\right)\right) - c_1\sum_{k\in B} e_1^k - c_2\sum_{k\in B} e_2^k \qquad (26)\\
&\quad= \sum_{k\in B} v^k - c_1\sum_{k\in B} e_1^k - c_2\sum_{k\in B} e_2^k.
\end{aligned}$$

Hence, by (24), (25), and (26),

$$\mathit{\Pi}_1(\boldsymbol{e}_1, \boldsymbol{e}_2) + \mathit{\Pi}_2(\boldsymbol{e}_1, \boldsymbol{e}_2) = \mathit{\Pi}_1\left(\frac{c_2}{c_1}\boldsymbol{e}_2, \boldsymbol{e}_2\right) + \mathit{\Pi}_2\left(\boldsymbol{e}_1, \frac{c_1}{c_2}\boldsymbol{e}_1\right) \tag{27}$$

$$\mathit{\Pi}_1(\boldsymbol{e}_1, \boldsymbol{e}_2) + \mathit{\Pi}_2(\boldsymbol{e}_1, \boldsymbol{e}_2) = \mathit{\Pi}_1\left(\boldsymbol{e}_1, \frac{c_1}{c_2}\boldsymbol{e}_1\right) + \mathit{\Pi}_2\left(\frac{c_2}{c_1}\boldsymbol{e}_2, \boldsymbol{e}_2\right). \tag{28}$$

Now, suppose that $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ is a Nash equilibrium of the model. Then

$$\mathit{\Pi}_1(\boldsymbol{e}_1, \boldsymbol{e}_2) \geq \mathit{\Pi}_1\left(\frac{c_2}{c_1}\boldsymbol{e}_2, \boldsymbol{e}_2\right) \tag{29}$$

and

$$\mathit{\Pi}_2(\boldsymbol{e}_1, \boldsymbol{e}_2) \geq \mathit{\Pi}_2\left(\boldsymbol{e}_1, \frac{c_1}{c_2}\boldsymbol{e}_1\right). \tag{30}$$

By (27) and (29)

$$\mathit{\Pi}_2(\boldsymbol{e}_1, \boldsymbol{e}_2) \leq \mathit{\Pi}_2\left(\boldsymbol{e}_1, \frac{c_1}{c_2}\boldsymbol{e}_1\right).$$

This, together with (30) yields

$$\mathit{\Pi}_2(\boldsymbol{e}_1, \boldsymbol{e}_2) = \mathit{\Pi}_2\left(\boldsymbol{e}_1, \frac{c_1}{c_2}\boldsymbol{e}_1\right) \tag{31}$$

and further, by (27),

$$\mathit{\Pi}_1(\boldsymbol{e}_1, \boldsymbol{e}_2) = \mathit{\Pi}_1\left(\frac{c_2}{c_1}\boldsymbol{e}_2, \boldsymbol{e}_2\right). \tag{32}$$

It follows that, for all $i \in \{1,2\}$,

$$\Pi_i(\boldsymbol{e}_1, \boldsymbol{e}_2) = p_i\left(1, \frac{c_1}{c_2}\right) \mathbf{1}^T \boldsymbol{v} - c_{-i}\mathbf{1}^T \boldsymbol{e}_{-i}. \tag{33}$$

Given $i \in \{1,2\}$ and $j \in \{1,2\}$, let

$$D_j p_i = \frac{\partial p_i(x_1, x_2)}{\partial x_j}$$

denote the derivative of p_i with respect to the j'th argument.

Since

$$\Pi_1(\boldsymbol{e}_1, \boldsymbol{e}_2) = \Pi_1\left(\frac{c_2}{c_1}\boldsymbol{e}_2, \boldsymbol{e}_2\right)$$

and $\boldsymbol{e}_1$ is a best response to $\boldsymbol{e}_2$ so $(c_2/c_1)\boldsymbol{e}_2$ is also a best response to $\boldsymbol{e}_2$. Hence, by the first order condition,

$$\begin{aligned} D_1p_1\left(\frac{c_2}{c_1}y_2^k, y_2^k\right)v^k + \sum_{l \in B\setminus\{k\}} \rho_{kl} D_1 p_1\left(\frac{c_2}{c_1}y_2^k, y_2^k\right)v^l - c_1 = 0, \text{ if } e_2^k > 0,\\ D_1p_1\left(\frac{c_2}{c_1}y_2^k, y_2^k\right)v^k + \sum_{l \in B\setminus\{k\}} \rho_{kl} D_1 p_1\left(\frac{c_2}{c_1}y_2^k, y_2^k\right)v^l - c_1 \le 0, \text{ if } e_2^k = 0, \end{aligned} \tag{34}$$

Defining, for $k \in B$,

$$\mu_1^k = D_1p_1\left(\frac{c_2}{c_1}y_2^k, y_2^k\right)\frac{v^k}{c_1}$$

and dividing both sides by c_1, in matrix notation, (34) can be rewritten as

$$(\mathbf{I} + \boldsymbol{\rho})\boldsymbol{\mu}_1 = (\mathbf{1} - \boldsymbol{s}_1)$$

where, for any $k \in B$, $s_1^k \ge 0$, and $s_1^k = 0$ when $e_2^k > 0$. From this it follows that

$$\mathbf{1}^T\boldsymbol{e}_2 = (\mathbf{1} - \boldsymbol{s}_1)^T\boldsymbol{e}_2 = \boldsymbol{\mu}_1^T\left(\mathbf{I} + \boldsymbol{\rho}^T\right)\boldsymbol{e}_2 = \boldsymbol{\mu}_1^T\boldsymbol{y}_2 \tag{35}$$

Since p_1 is homogeneous of degree 0 so, for any $x_2 > 0$,

$$D_1p_1(x_1, x_2) = \frac{\partial p_1\left(\frac{x_1}{x_2}, 1\right)}{\partial x_1} = \frac{1}{x_2}D_1p_1\left(\frac{x_1}{x_2}, 1\right). \tag{36}$$

Hence, for any $k \in B$,

$$\mu_1^k = \frac{1}{y_2^k}D_1p_1\left(\frac{c_2}{c_1}, 1\right)\frac{v^k}{c_1}$$

and

$$\boldsymbol{\mu}_1^T \boldsymbol{y}_2 = \frac{1}{c_1} D_1 p_1 \left(\frac{c_2}{c_1}, 1 \right) \mathbf{1}^T \boldsymbol{v} = D_1 p_1 (c_2, c_1) \, \mathbf{1}^T \boldsymbol{v}.$$

Combining this, (33) and (35) we get

$$\mathit{\Pi}_1 \left(\boldsymbol{e}_1, \boldsymbol{e}_2 \right) = \left(p_1(c_2, c_1) - c_2 D_1 p_1(c_2, c_1) \right) \mathbf{1}^T \boldsymbol{v}.$$

By analogous derivation, using the fact that $(c_1/c_2)\boldsymbol{e}_1$ is a best response to $\boldsymbol{e}_1$, we also obtain

$$\mathit{\Pi}_2 \left(\boldsymbol{e}_1, \boldsymbol{e}_2 \right) = \left(p_2(c_2, c_1) - c_1 D_2 p_2(c_2, c_1) \right) \mathbf{1}^T \boldsymbol{v}.$$

This characterizes the equilibrium payoffs of the two players.

For the remaining part of the proof, suppose that $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ is an interior Nash equilibrium of the conflict game, i.e. for all $i \in \{1, 2\}$ and all $k \in B$, $e_i^k > 0$. Since $\boldsymbol{e}_i$ is a best response to $\boldsymbol{e}_{-i}$ (for $i \in \{1, 2\}$,) and since $\boldsymbol{e}_{-i} > \mathbf{0}$ so, for any $i \in \{1, 2\}$,

$$D_i p_i \left(y_i^k, y_{-i}^k \right) v^k + \sum_{l \in B \setminus \{k\}} \rho_{kl} D_i p_i \left(y_i^k, y_{-i}^k \right) v^l - c_i = 0. \tag{37}$$

Defining, for $k \in B$,

$$\mu_i^k = D_i p_i \big(y_1^k, y_2^k \big) \, \frac{v^k}{c_i}$$

and dividing both sides by c_i, in matrix notation, (37) can be rewritten as

$$(\mathbf{I} + \boldsymbol{\rho}) \boldsymbol{\mu}_i = \mathbf{1}. \tag{38}$$

Since $(\mathbf{I} + \boldsymbol{\rho})$ is invertible, there is a unique $\boldsymbol{\mu}_i$ solving this equation. Hence, $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$ and, consequently,

$$\frac{D_1 p_1 \big(y_1^k, y_2^k \big)}{D_2 p_2 \big(y_1^k, y_2^k \big)} = \frac{c_1}{c_2}, \tag{39}$$

On the other hand, by the assumption of homogeneity of degree zero, from Euler's homogeneous function theorem, for all $k \in B$, $i \in \{1, 2\}$, $y_1^j \geq 0$, and $y_2^j \geq 0$,

$$y_1^k D_1 p_1 (y_1^k, y_2^k) + y_2^k D_2 p_1 (y_1^k, y_2^k) = 0. \tag{40}$$

Furthermore, by the no-tie property,

$$D_2 p_1 \big(y_1^k, y_2^k \big) = \frac{\partial \left(1 - p_2 \big(y_1^k, y_2^k \big) \right)}{\partial y_2^k} = - D_2 p_2 \big(y_1^k, y_2^k \big) \,. \tag{41}$$

Combing (40) and (41), we get

$$\frac{D_1 p_1 (y_1^k, y_2^k)}{D_2 p_2 (y_1^k, y_2^k)} = \frac{y_2^k}{y_1^k}. \tag{42}$$

Combining (39) and (42) yields, at $(\boldsymbol{e}_1, \boldsymbol{e}_2)$,

$$\frac{y_2^k}{y_1^k} = \frac{c_1}{c_2}. \tag{43}$$

which, by homogeneity of degree 0, determines the equilibrium winning probability:

$$p_i\big(y_1^k, y_2^k\big) = p_i\left(1, \frac{y_2^k}{y_1^k}\right) = p_i\left(1, \frac{c_1}{c_2}\right) = p_i(c_2, c_1)\,.$$

By (38), the expected equilibrium effort satisfies

$$\mathbf{1}^T \boldsymbol{e}_i = \boldsymbol{\mu}_i^T \left(\mathbf{I} + \boldsymbol{\rho}^T\right) \boldsymbol{e}_i = \boldsymbol{\mu}_i^T \boldsymbol{y}_i.$$

By (36), for any $j \in B$ and $i \in \{1, 2\}$,

$$\mu_i^k = \frac{1}{y_i^k} D_i p_i \left(\frac{c_2}{c_1}, 1\right) \frac{v^j}{c_i} = \frac{1}{y_i^k} D_i p_i(c_2, c_1)\, v^j.$$

Hence the expected total expenditure of player i at $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ is

$$c_i \mathbf{1}^T \boldsymbol{e}_i = c_i D_i p_i(c_2, c_1)\, \mathbf{1}^T \boldsymbol{v}.$$

□

A.3 Vertex index

In this section we prove Propositions 2 and 3, tying equilibrium efforts of the players to a centrality index in the network.

Proof of Proposition 2. By the analysis of equilibrium real efforts profile in proof of Theorem 1, strategy profile $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ is a Nash equilibrium of the game if and only if, for all $i \in \{1, 2\}$,

$$\boldsymbol{e}_i = \frac{\gamma c_1^\gamma c_2^\gamma}{c_i \left(c_1^\gamma + c_2^\gamma\right)^2} \boldsymbol{x} \tag{44}$$

where $\boldsymbol{x}$ satisfies

$$\begin{aligned} \left(\mathbf{I} + \boldsymbol{\rho}^T\right) \boldsymbol{x} &= \boldsymbol{v} \oslash \boldsymbol{\mu} = \mathrm{diag}(\boldsymbol{v}) \mathbf{1} \oslash \boldsymbol{\mu} \\ \boldsymbol{x} &\geq \mathbf{0} \end{aligned}$$

and $\boldsymbol{\mu}$ satisfies

$$\begin{aligned} \left(\mathbf{I} + \boldsymbol{\rho}\right) \boldsymbol{\mu} &\leq \mathbf{1}, \\ \boldsymbol{\mu} &> \mathbf{0}, \end{aligned}$$

where the first constraint holds with equality if $x_i > 0$.

Thus vector $\boldsymbol{x}$ satisfies the constraints above if and only if it satisfies

$$\begin{aligned}(\mathbf{I}+\boldsymbol{\rho})\,\mathrm{diag}(\boldsymbol{v})\mathbf{1} \oslash \left(\left(\mathbf{I}+\boldsymbol{\rho}^T\right)\boldsymbol{x}\right) &\le \mathbf{1},\\ \boldsymbol{x} &\ge \mathbf{0},\end{aligned}$$

where the first constraint holds with equality when $x_i > 0$. This is equivalent to

$$\begin{aligned}(\mathbf{I}+\boldsymbol{\rho})\,\mathrm{diag}(\boldsymbol{v})\mathbf{1} \oslash \left(\left(\mathbf{I}+\boldsymbol{\rho}^T\right)\boldsymbol{x}\right) &\le \mathbf{1},\\ \boldsymbol{x}^T\left(\mathbf{1}-(\mathbf{I}+\boldsymbol{\rho})\,\mathrm{diag}(\boldsymbol{v})\mathbf{1} \oslash \left(\left(\mathbf{I}+\boldsymbol{\rho}^T\right)\boldsymbol{x}\right)\right) &= 0,\\ \boldsymbol{x} &\ge \mathbf{0}.\end{aligned} \tag{45}$$

Since

$$\boldsymbol{x}^T(\mathbf{I}+\boldsymbol{\rho})\,\mathrm{diag}(\boldsymbol{v})\mathbf{1} \oslash \left(\left(\mathbf{I}+\boldsymbol{\rho}^T\right)\boldsymbol{x}\right) = \mathbf{1}^T\boldsymbol{v}$$

so the equality constraint is equivalent to $\mathbf{1}^T\boldsymbol{x} = \mathbf{1}^T\boldsymbol{v}$ and so $\boldsymbol{x}$ satisfies (45) if and only if it satisfies

$$\begin{aligned}\mathbf{1}^T\boldsymbol{x} &= \mathbf{1}^T\boldsymbol{v}\\ (\mathbf{I}+\boldsymbol{\rho})\,\mathrm{diag}(\boldsymbol{v})\mathbf{1} \oslash \left(\left(\mathbf{I}+\boldsymbol{\rho}^T\right)\boldsymbol{x}\right) &\le \mathbf{1},\\ \boldsymbol{x} &\ge \mathbf{0}.\end{aligned} \tag{46}$$

Thus we have shown that strategy profile $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ is a Nash equilibrium of the game if and only if, for all $i \in \{1, 2\}$, it satisfies (44) where $\boldsymbol{x}$ satisfies (46). By Theorem 1, $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ is a unique equilibrium. Therefore $\boldsymbol{x}$ uniquely satisfies (46).

In the last step of the proof we will show that the solution of the optimization problem

$$\begin{aligned}\min_{\boldsymbol{z}\in\mathbb{R}^B}\ & \mathbf{1}^T\boldsymbol{z} \text{ s.t.}\\ & (\mathbf{I}+\boldsymbol{\rho})\mathrm{diag}(\boldsymbol{v})\mathbf{1} \oslash \left(\left(\mathbf{I}+\boldsymbol{\rho}^T\right)\boldsymbol{z}\right) \le \mathbf{1}\\ & \boldsymbol{z} \ge \mathbf{0}.\end{aligned} \tag{47}$$

is equal to $\mathbf{1}^T\boldsymbol{v}$. For assume otherwise. Then the solution W of (47) satisfies $W < V = \mathbf{1}^T\boldsymbol{v}$. Let $\boldsymbol{z}$ be a minimizer of (47). We will construct two distinct vectors $\boldsymbol{x}'$ and $\boldsymbol{x}''$ that satisfy (46). Take any battlefield $i \in B$ and battlefield $j \in B$ such that $i \neq j$ (since $|B| \geq 2$ such battlefields exist). Let $x'_i = z_i + V - W$ and $\boldsymbol{x}'_{-i} = \boldsymbol{z}_{-i}$. Similarly, let $x''_j = z_j + V - W$ and $\boldsymbol{x}''_{-j} = \boldsymbol{z}_{-j}$. Clearly $\boldsymbol{x}' \neq \boldsymbol{x}''$. Moreover, by construction, $\mathbf{1}^T\boldsymbol{x}' = \mathbf{1}^T\boldsymbol{x}'' = \mathbf{1}^T\boldsymbol{v}$ and, since $\boldsymbol{z} \geq \mathbf{0}$ so $\boldsymbol{x}' \geq \mathbf{0}$ and $\boldsymbol{x}'' \geq \mathbf{0}$. We will show that both $\boldsymbol{x}'$ and $\boldsymbol{x}''$ satisfy the first inequality constraint of (47). Since $\boldsymbol{z}$ satisfies the constraint so, for all $k \in B$,

$$\frac{v^k}{z_k + \sum_{r\in B\setminus\{k\}}\rho_{rk}z_r} + \sum_{l\in B\setminus\{k\}}\frac{v^l\rho_{kl}}{z_l + \sum_{r\in B\setminus\{l\}}\rho_{rl}z_r} \le 1.$$

Since $\boldsymbol{\rho} \geq \mathbf{0}$, $\boldsymbol{v} > 0$, $\boldsymbol{x}' \geq \boldsymbol{z}$, and $\boldsymbol{x}'' \geq \boldsymbol{z}$, so the inequality is also satisfied by $\boldsymbol{x}'$ and $\boldsymbol{x}''$, for all $k \in B$. Hence $\boldsymbol{x}'$ and $\boldsymbol{x}''$ both satisfy the first inequality constraint of (47). Thus we have shown that they both satisfy (46). Since they are distinct, this contradicts uniqueness of the solution to (46). Therefore the solution to (47) must be equal to $\mathbf{1}^T\boldsymbol{v}$. By uniqueness of the solution to (46), the minimizer of (46) must be unique. Hence a strategy profile $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ is a Nash equilibrium of the game if and only if it satisfies (44) and (47). This completes the proof. □

Proof of Proposition 3. Let $\boldsymbol{x}$ be a vector solving (9). That $P(\boldsymbol{x})$ is an out-dominating set of $\boldsymbol{\rho}$ follows directly from Propositions 5 and 2. Point 1 follows directly from the fact that $P(\boldsymbol{x})$ is an out-dominating set in $\boldsymbol{\rho}$.

For points 2 and 3, let $\tilde{\boldsymbol{\rho}} = \mathbf{I} + \boldsymbol{\rho}$ and, for any $k \in B$ and $\boldsymbol{x} \in \mathbb{R}^B$, let

$$\varphi_k(\boldsymbol{x}) = \frac{v^k}{x_k + \sum_{r \in B\setminus\{k\}} \rho_{lk} x_l} + \sum_{l \in B\setminus\{k\}} \frac{v^l \rho_{kl}}{x_l + \sum_{r \in B\setminus\{l\}} \rho_{rl} x_r} = \sum_{l \in B} \frac{v^l \tilde{\rho}_{kl}}{\sum_{r \in B} \tilde{\rho}_{rl} x_r}$$

so that the inequality constraints

$$(\mathbf{I} + \boldsymbol{\rho})\mathrm{diag}(\boldsymbol{v})\left(\mathbf{1} \oslash \left(\left(\mathbf{I} + \boldsymbol{\rho}^T\right)\boldsymbol{z}\right)\right) \leq \mathbf{1}$$

of (9) can be written as $\varphi_k(\boldsymbol{z}) \leq 1$, for all $k \in B$. Notice that, for $i \in B$,

$$D_i\varphi_k = \frac{\partial \varphi_k(\boldsymbol{x})}{\partial x_i} = -\sum_{l \in B} \frac{v^l \tilde{\rho}_{kl} \tilde{\rho}_{il}}{\left(\sum_{r \in B} \tilde{\rho}_{rl} x_r\right)^2}.$$

Let $S = \{\boldsymbol{z} \in \mathbb{R}_{\geq 0} : \text{for all } l \in B,\ \sum_{r \in B} \tilde{\rho}_{rl} z_r > 0\}$ be the set of all vectors $\boldsymbol{z}$ such that $P(\boldsymbol{z})$ is an out-dominating set in $\boldsymbol{\rho}$.

For point 2, suppose that $\boldsymbol{x}$ solves the optimization problem (9). Then, for any $i \in B$ and $j \in B$ with $x_i > x_j$, there exists $k \in B$ such that $D_i\varphi_k(\boldsymbol{x}) < D_j\varphi_k(\boldsymbol{x})$. For assume otherwise. Then there exist $i \in B$ and $j \in B$ with $x_i > x_j$ such that for all $k \in B$, $D_j\varphi_k(\boldsymbol{x}) \leq D_i\varphi_k(\boldsymbol{x})$. Since $\boldsymbol{x} \in S$ so for all $k \in B$, φ_k is continuous and continuously differentiable in the neighbourhood of $\boldsymbol{x}$. Moreover, there exist $\varepsilon \in (0, x_i)$ such that decreasing x_i by ε and increasing x_j by ε weakly reduces the values of $\varphi_k(\boldsymbol{x})$, for all $k \in B$. Hence this modification maintains the constraints of the optimization problem. Moreover, it maintains the value of the objective function. Hence there is another vector $\boldsymbol{x}'$ that solves the optimization problem (9), a contradiction to uniqueness of the solution to (9) stated in Proposition 2.

If node $i \in B$ is dominated by node $j \in B$ then $\tilde{\rho}_{jl} \geq \tilde{\rho}_{il}$ for all $l \in B$. Hence, for any $\boldsymbol{z} \in S$ and any $k \in B$, $D_j\varphi_k(\boldsymbol{z}) \leq D_i\varphi_k(\boldsymbol{z})$. By the claim above, if $\boldsymbol{x}$ solves (9) then $x_j \geq x_i$.

For point 3, suppose that $\boldsymbol{x}$ solves the optimization problem (9). Then, for any $i \in B$ with $x_i > 0$ and any $j \in B$, if there exists $k \in B$ such that $D_j\varphi_k(\boldsymbol{x}) < D_i\varphi_k(\boldsymbol{x})$ then there exists $l \in B$ such that $D_j\varphi_l(\boldsymbol{x}) \geq D_i\varphi_l(\boldsymbol{x})$ and $D_i\varphi_l(\boldsymbol{x}) < 0$. For assume otherwise. Then there exist $i \in B$ with $x_i > 0$ and $j \in B$ such that for all $k \in B$, either $D_j\varphi_k(\boldsymbol{x}) < D_i\varphi_k(\boldsymbol{x})$ or $D_i\varphi_k(\boldsymbol{x}) \geq 0$, and $D_j\varphi_k(\boldsymbol{x}) < D_i\varphi_k(\boldsymbol{x})$ for at least one $k \in B$. Since $\boldsymbol{x} \in S$ so for all $k \in B$, φ_k is continuous and continuously differentiable in the neighbourhood of $\boldsymbol{x}$. Moreover, there exist $\varepsilon \in (0, x_i)$ and $\varepsilon' \in (0, \varepsilon)$ such that decreasing x_i by ε and increasing x_j by ε' reduces the values of $\varphi_k(\boldsymbol{x})$, for all $k \in B$. Hence this modification maintains the constraints of the optimization problem. Moreover, it decreases the value of the objective function, a contradiction with the assumption that $\boldsymbol{x}$ solves the optimization problem (9).

If node $i \in B$ is strictly dominated by node $j \in B$ then $\tilde{\rho}_{jl} > \tilde{\rho}_{il}$ for all $l \in B$ such that $\tilde{\rho}_{il} > 0$. Hence, for any $\boldsymbol{z} \in S$ and any $k \in B$, either $D_j\varphi_k(\boldsymbol{z}) < D_i\varphi_k(\boldsymbol{z})$ or $D_i\varphi_k(\boldsymbol{x}) \geq 0$ and $D_j\varphi_i(\boldsymbol{z}) < D_i\varphi_i(\boldsymbol{z})$ (so there exists $k \in B$ such that $D_j\varphi_k(\boldsymbol{x}) < D_i\varphi_k(\boldsymbol{x})$). By the claim above, if $\boldsymbol{x}$ solves (9) then $x_j = 0$. $\square$