# Nonparametric spectral-based estimation of latent structures

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# Paper question

- Economist like unobserved heterogeneity and dynamic factor models.
- Usually discrete mixtures of parametric distributions (derived from theory)
- For identification and also estimation, it is useful to consider discrete mixtures of nonparametric models.
- This paper proposes a simple estimation procedure for discrete mixtures and hidden Markov models of nonparametric distribution components.

# Identification

- The question of identification in latent structures is the topic of a very recent and active literature.
- Nonparametric identification from univariate/cross-sectional data typically fails. (Some exceptions for location models)
- Multivariate data (panel data) can present a powerful identification source.
    1. Finite mixtures/latent-class models: Hall and Zhou (2003); Allman et al. (2009)
    2. (Dynamic) discrete-choice models: Magnac and Thesmar (2002); Kasahara and Shimotsu (2009)
    3. Hidden Markov/regime-switching models: Allman et al. (2009); Gassiat et al. (2013)
    4. Models for corrupted and misclassified data: Schennach (2004); Hu and Schennach (2008)

# Contribution

We propose a new constructive identification argument...

that delivers a least square-type estimator for mixture weights...

allowing for asymptotic distributional theory.

# Discrete mixtures of discrete distributions

- Let $(y_1,...,y_q)$ be $q$ discrete variables with supp$(y_i) = \{1,...,\kappa_i\}$.
- There exists a latent variable $x \in \{1,...,r\}$ with $\pi_j \equiv \Pr\{x = j\}$.
- Let $p_{ij} \in [0,1]^{\kappa_i}$ denote the vector of conditional probability masses of $y_i$ given $x = j$:

$$p_{ij}(k) \equiv \Pr\{y_i = k | x = j\}, \quad k = 1,...,\kappa_i$$

# Unconditional distribution for DMs

- The unconditional joint PDF of $(y_1, \ldots, y_q)$ is

$$\mathbb{P}(y_1, \ldots, y_q) = \sum_{j=1}^{r} \pi_j p_{1j}(y_1) p_{2j}(y_2) \ldots p_{qj}(y_q)$$

- The set of values $\mathbb{P}(y_1, \ldots, y_q)$ for all $(y_1, \ldots, y_q)$ defines a $q$-dimensional array

$$\mathbb{P} = \sum_{j=1}^{r} \pi_j p_{1j} \otimes p_{2j} \otimes \cdots \otimes p_{qj}$$

  - $\otimes$ is the Kronecker product

# Hidden Markov models

- There are $q$ discrete latent variables $(x_1, \ldots, x_q)$ for $q$ measurements $(y_1, \ldots, y_q)$.
- **Stationarity:**

$$\begin{aligned} \Pr\{x_i = j\} &= \pi_j, \quad i = 1, \ldots, q \\ \Pr\{x_{i+1} | x_i\} &= K(x_i, x_{i+1}), \quad i = 1, \ldots, q-1 \\ \Pr\{y_i = k | x_i = j\} &= p_j(k), \quad k = 1, \ldots, \kappa \end{aligned}$$

- **Conditional independence:** measurements $y_1, \ldots, y_q$ are independent conditional on $(x_1, \ldots, x_q)$.

# Unconditional distribution for HHMs (1)

- The unconditional joint PDF of $(y_1, ..., y_3)$ is

$$\begin{aligned}
&\mathbb{P}(y_1, y_2, y_3) \\
&= \sum_{j_1=1}^{r} \left\{ \pi_{j_1} p_{j_1}(y_1) \sum_{j_2=1}^{r} \left[ K(j_1, j_2) p_{j_2}(y_2) \sum_{j_3=1}^{r} K(j_2, j_3) p_{j_3}(y_3) \right] \right\} \\
&= \sum_{j_2=1}^{r} \left\{ \left[ \sum_{j_1=1}^{r} p_{j_1}(y_1) \pi_{j_1} K(j_1, j_2) \right] p_{j_2}(y_2) \left[ \sum_{j_3=1}^{r} K(j_2, j_3) p_{j_3}(y_3) \right] \right\}
\end{aligned}$$

# Unconditional distribution for HHMs (2)

- Let $P = [p_1, \ldots, p_r] \in \mathbb{R}^{\kappa \times r}$ and $\Pi = \operatorname{diag}(\pi_1, \ldots, \pi_r)$.
- Hence the 3-dimensional array

$$\mathbb{P} = \sum_{j=1}^{r} (P\Pi K)_j \otimes p_j \otimes \left(PK^\top\right)_j$$

  where $M_j$ denotes the $j$th column of matrix $M$
- If $q > 3$ one can select all consecutive triples or regroup observations into 3 consecutive clusters: $(y_1, \ldots, y_{k-1}), y_k, (y_{k+1}, \ldots, y_q)$.

# Identification of such latent array structures

## Kruskal (Psychometrica 1976, Linear Algebra Appl. 1977)

- Consider a $\kappa_1 \times \kappa_2 \times \kappa_3$ array $\mathbb{P} = \sum_{j=1}^{r} p_{1j} \otimes p_{2j} \otimes p_{3j}$
- Let $P_i = [p_{i1}, \ldots, p_{ir}] \in \mathbb{R}^{\kappa_i \times r}, i = 1, 2, 3$
- Let $r_i = \max\{k$ : all collections of $k$ columns of $P_i$ are independent$\}$ (the Kruskal-rank of $P_i$).
  - Note that if $P \in \mathbb{R}^{\kappa \times r}$ has rank $r$ it also has Kruskal-rank $r$.
- If $r_1 + r_2 + r_3 \geq 2r + 2$ then $\mathbb{P}$ uniquely determines the matrices $P_i$ up to simultaneous column-permutation and common column-scaling.

# Application to statistics

## Allman, Matias and Rhodes (AoS, 2009)

- Allman et al. use Kruskal's result to give conditions for the identification of discrete mixtures of discrete and continuous nonparametric distributions, hidden Markov models and some stochastic graphs.

- Kruskal's theorem applies with

$$\mathbb{P} = \sum_{j=1}^{r} \pi_j p_{1j} \otimes p_{2j} \otimes p_{3j}$$

$$P_1 = [\pi_1 p_{11}, \ldots, \pi_r p_{1r}], P_i = [p_{i1}, \ldots, p_{ir}], i > 1$$

- (Corollary 2) Since $\text{sum}(P_1, 1) = [\pi_1, \ldots, \pi_r]$ and $\text{sum}(P_i, 1) = [1, \ldots, 1], i > 1$, then, if $r_1 + r_2 + r_3 \geq 2r + 2$, group-probabilities $\pi_j$ and conditional probabilities $p_{ij}$ are identified up to labeling.
- (Theorem 8) Holds for continuous mixture components if the component densities are linearly independent ($r_1 = r_2 = r_3 = r$).

- The parameters of an HMM with $r$ hidden states and $\kappa$ observable states are generically identifiable from the marginal distribution of $2k+1$ consecutive variables provided $k$ satisfies

$$\binom{k+\kappa-1}{\kappa-1} \geq r$$

- Note that $\binom{k+\kappa-1}{\kappa-1} = \kappa$ for $k=1$ (3 measurements) and $\binom{k+\kappa-1}{\kappa-1} = k+1$ for $\kappa=2$ (binary outcomes).

# Application to HMMs

## Gassiat, Cleynen, Robin (arXiv, 2013), Theorem 2.1

- They use Allman et al.'s result to prove the following result.
- Assume that $r$ is known, $P = [p_1, ..., p_r]$ is full column rank, and $K$ has full rank. Then $K$ and $P$ are identifiable from from the distribution of 3 consecutive observations $(y_1, y_2, y_3)$ up to label swapping of the hidden states.
- Estimation by penalized ML or EM algorithm.

# Constructive identification procedures

- There exists few constructive identification procedures.
- De Lathauwer (SIAM, 2006) applies to the case where one outcome (say $y_1$) is such that $P_1$ is full column rank.
- However it provides identification only up to relabeling AND scaling.
- Group probabilities $\pi_j$ are thus not identified.
- We propose one such constructive identification that works both for DMs and HMMs, inspired from ICA or blind deconvolution.

# DMs

- $\mathbb{P} = \sum_{j=1}^{r} \pi_j p_{1j} \otimes p_{2j} \otimes p_{3j}$
- Let $\Pi = \mathrm{diag}(\pi_1, \ldots, \pi_r)$, and
  $P_i = [p_{i1}, \ldots, p_{ir}] \in \mathbb{R}^{\kappa_i \times r}, i = 1, 2, 3.$
- Assume $\mathrm{rank}(P_i) = r$ and $\pi_j > 0$.

# DMs

- $P_1 \Pi P_2^\top = \sum_{j=1}^r \pi_j p_{1j} p_{2j}^\top = \sum_{j=1}^r \pi_j p_{1j} \otimes p_{2j}$ is the matrix containing probabilities $\mathbb{P}(y_1, y_2)$. Observable.
- SVD on $P_1 \Pi P_2^\top$, which has rank $r$, allows to construct $U$ and $V$ such that

$$\underset{r\times\kappa_1}{U} P_1 \Pi P_2^\top \underset{\kappa_2\times r}{V^\top} = I_r \Rightarrow (VP_2)^\top = (UP_1\Pi)^{-1} \equiv \underset{r\times r}{Q^{-1}}$$

- $\mathbb{P}(:,:,k) = \sum_{j=1}^r \pi_j p_{1j} \otimes p_{2j} \otimes p_{3j}(k) = P_1 \Pi D_{3k} P_2^\top$, with $D_{3k} = \mathrm{diag}[p_{31}(k), ..., p_{3r}(k)]$, is the matrix containing probabilities $\mathbb{P}(y_1, y_2, k)$ (for any $y_1, y_2$ and $y_3 = k$). Also observable.
- $W_k = U\mathbb{P}(:,:,k)V^\top = QD_{3k}Q^{-1}$ (whitening)
- $P_3$ identified by the eigenvalues of matrices $W_1, ..., W_{\kappa_3}$
- Repeat for $P_1$ and $P_2$.
- $\pi = [\pi_1; ...; \pi_r]$ identified from $\mathbb{P}(y_i) = \sum_{j=1}^r \pi_j p_{ij}(y_i) = P\pi$

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# HMMs

- $\mathbb{P} = \sum_{j=1}^{r} (P\Pi K)_j \otimes p_j \otimes \left(PK^\top\right)_j, \quad \Pi = \mathrm{diag}(\pi_1, \ldots, \pi_r)$
- Assume $K$ full rank, $P = [p_1, \ldots, p_r] \in \mathbb{R}^{\kappa \times r}$ full column rank and $\pi_j > 0$.

# HMMs

- One can put all $\mathbb{P}(y_1, y_2, y_3)$ for fixed $y_2 \in \{1, ..., \kappa\}$ in the matrix $\mathbb{P}(:, k, :) = P\Pi K D_{2k} K P^\top$, $\quad D_{2k} = \mathrm{diag}(p_1(k), ..., p_r(k))$
- Note that the matrix $P\Pi K^2 P^\top$ is the matrix containing probabilities $\mathbb{P}(y_1, y_3)$.
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# Estimation procedure

- Matrices $W_k$ thus have to be simultaneously diagonalized.
- Approximate joint diagonalization by least squares:

$$Q = \arg\min_{Q} \sum_{k=1}^{\kappa_3} \left\| W_k - Q D_k Q^{-1} \right\|_F^2, \qquad D_k \equiv \mathrm{diag}\left[ Q^{-1} W_k Q \right]$$

- Algorithm in Iferroudjene, Abed-Meraim and Belouchrani (Applied Math. and Computation, 2009)
- Advantage of LS: asymptotic theory is possible

# Continuous outcomes

- Requires discretization
- We use orthogonal polynomials (Chebychev)

# Discrete mixtures of continuous distributions

- Conditional PDF of $y_i$ given $= j$:

$$f_{ij}(y) \simeq \sum_{k=1}^{\kappa_i} p_{ij}(k)\varphi_k(y), \quad p_{ij}(k) = \int_{-1}^{1} \varphi_\kappa(u) f_{ij}(u)\,\mathrm{d}u$$

- $(\varphi_k)$ complete orthonormal set of functions:

$$\int \varphi_k(y)\varphi_\ell(y)\rho(y)\,\mathrm{d}y = \delta_{k\ell}$$

- Three observations:

$$\begin{aligned} f(y_1, y_2, y_3) &= \sum_{j=1}^{r} \pi_j f_{1j}(y_1) f_{2j}(y_2) f_{3j}(y_3) \\ &\simeq \sum_{j=1}^{r} \pi_j p_{1j} \otimes p_{2j} \otimes p_{3j} \end{aligned}$$

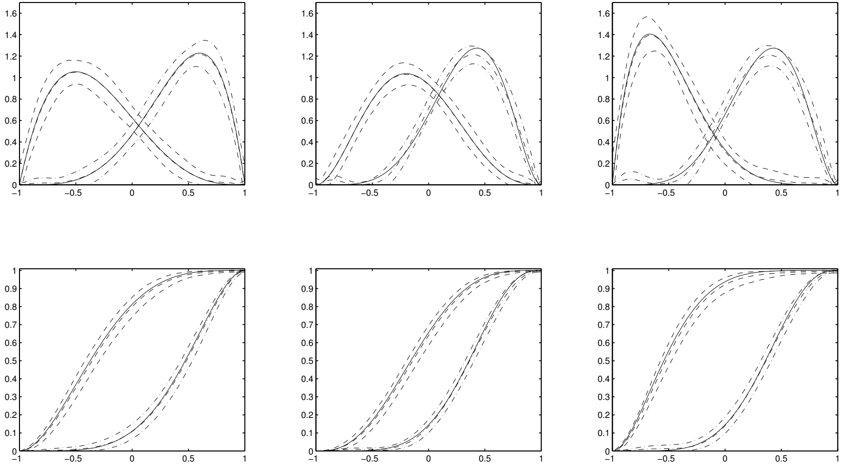
- Note that $\mathrm{sum}(p_{ij}) \neq 1$. Yet the identification algorithm continues to work.

# Asymptotic theory

- Standard convergence rates because the weights are root-$n$ consistent
- Extends to hidden Markov models for continuous outcomes

- We generate data from a heterogenous mixture of beta distributions on $[-1, 1]$
- $r = 2$; $q = 3$; $\pi_1 = \pi_2 = \frac{1}{2}$
- Chebychev polynomials of the first kind for $\phi_i$.
- Orthogonal-series estimators are not bona fide. Adjust estimates ex post via Gajek's (1986) projection procedure.

## $n = 500$



# Proportions

| | $n = 500$ | | | | $n = 1000$ | | | |
|---|---|---|---|---|---|---|---|---|
| | mean | | std | | mean | | std | |
| | $\pi_1$ | $\pi_2$ | $\pi_1$ | $\pi_2$ | $\pi_1$ | $\pi_2$ | $\pi_1$ | $\pi_2$ |
| $i = 1$ | .5133 | .4794 | .0257 | .0260 | .5090 | .4869 | .0186 | .0186 |
| $i = 2$ | .5130 | .4854 | .0300 | .0301 | .5092 | .4895 | .0204 | .0205 |
| $i = 3$ | .4978 | .4948 | .0319 | .0320 | .4980 | .4989 | .0231 | .0229 |

# Example: HMMs

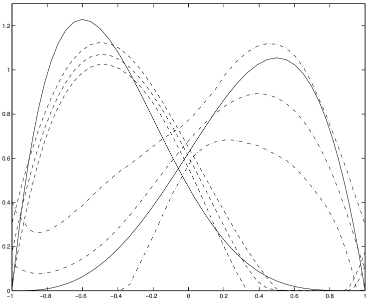
- Stationary probit model for a binary state variable

$$s_t = 1\{s_{t-1} \geq \varepsilon_t\}, \qquad \varepsilon_t \sim \mathscr{N}(0,1),$$

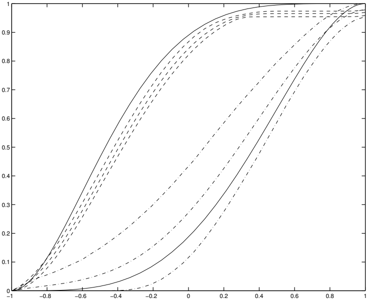
and suppose that,

$$f(y_t|s_t=0) \sim \text{left-skewed Beta}, \qquad f(y_t|s_t=1) \sim \text{right-skewed Beta}$$

- Steady state gives $\Pr[s_t = 0] \approx \frac{1}{4}$ and $K(0,0) = \frac{1}{2}$, $K(1,0) \approx \frac{1}{6}$.
- Most draws are from dominant regime ($s_t = 1$).



1.2
1
0.8
0.6
0.4
0.2
0
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

# State process

| parameter | value | mean | std |
|---|---|---|---|
| $\Pr[s_t = 1]$ | .7591 | .7255 | .0755 |
| $\Pr[s_t = 0]$ | .2409 | .2554 | .0786 |
| $K(0,0)$ | .5000 | .5731 | .3056 |
| $K(0,1)$ | .5000 | .3913 | .3494 |
| $K(1,0)$ | .1587 | .1352 | .0587 |
| $K(1,1)$ | .8413 | .8500 | .0608 |